Lecture 2 S.O.D. under wall-crossing

Naive idea #2 X smooth proj. variety $X \hookrightarrow \mathbb{P}^N$

codim $X = 1 \Rightarrow X$ is a Fano visitor (Kiem-Kim-Lee-Lee)

codim $X > 1 \Rightarrow$ f.f. $D^b(X) \hookrightarrow D^b(\mathrm{Bl}_X \mathbb{P}^N)$ (Orlov)

$\mathrm{Bl}_X \mathbb{P}^N$ is rarely Fano but...

Question Can we arrange it to be log Fano?

Def Y is called log Fano (or Fano type) variety if there exists a $\mathbb{Q}$-divisor $\Delta \subset Y$ such that (Y, Δ) has klt singularities & $-(K_Y + \Delta)$ is ample

I don't know any counterexamples (yet ☺)

In fact, there are some interesting examples:

Th (Aravena) Let X be a K3 surface with $\mathrm{Pic}\, X = \mathbb{Z} \cdot \Lambda$

Define genus of X by formula $\Lambda^2 = 2g-2$

If $g \geq 3$ then $X \underset{|\Lambda|}{\hookrightarrow} \mathbb{P}^g$ and $\mathrm{Bl}_X \mathbb{P}^g$ is log Fano!

However, this is not proved by exhibiting a boundary divisor Δ. Instead, a much stronger result is proved, a complete description of the stable base locus decomposition

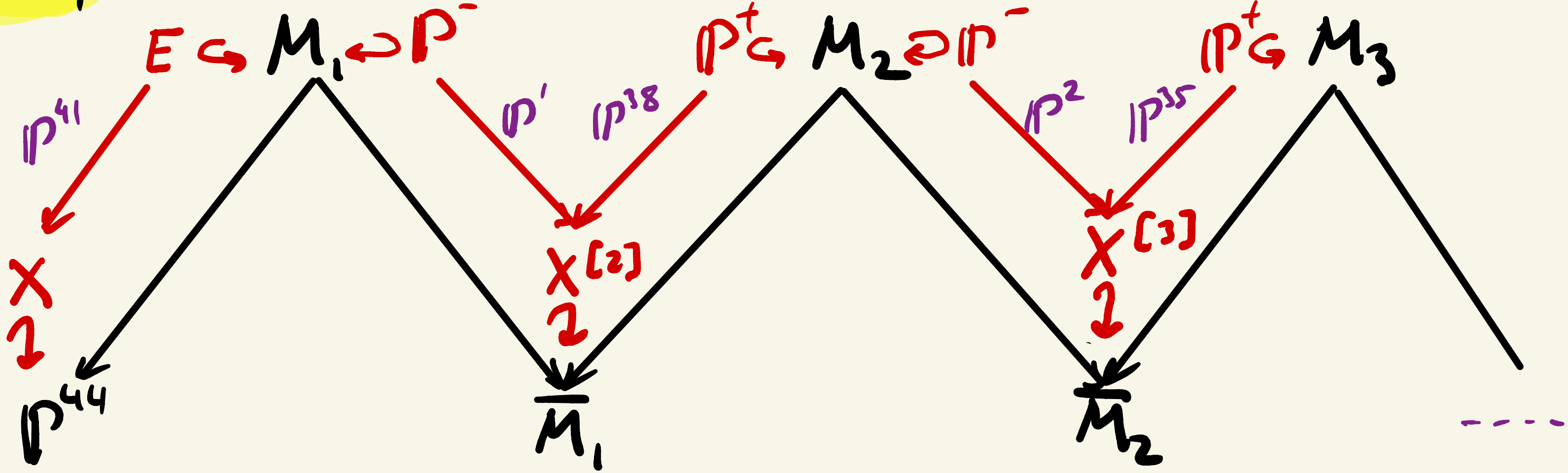
Example $g = 44$

Enter the genus g: 44

	c	d	mu	k^+	k^-	k^+-k^-	Mukai vector	Dim
0	0	-1	23	45	0	45	(1, 0, 1)	0
1	0	0	21	42	1	41	(1, 0, 0)	2
2	0	1	19	39	2	37	(1, 0, -1)	4
3	0	2	17	36	3	33	(1, 0, -2)	6
4	0	3	15	33	4	29	(1, 0, -3)	8
5	0	4	13	30	5	25	(1, 0, -4)	10
6	0	5	11	27	6	21	(1, 0, -5)	12
7	0	6	9	24	7	17	(1, 0, -6)	14
8	0	7	7	21	8	13	(1, 0, -7)	16
9	0	8	5	18	9	9	(1, 0, -8)	18
10	0	9	3	15	10	5	(1, 0, -9)	20
11	1	1	5/3	24	17	7	(3, -1, 14)	4
12	0	10	1	12	11	1	(1, 0, -10)	22
13	1	4	1	19	16	3	(3, -1, 13)	10
14	1	7	1/3	14	15	-1	(3, -1, 12)	16
15	2	2	1/5	18	21	-3	(5, -2, 34)	6

Any genus: https://colab.research.google.com/drive/1qUTYWFOgKur9JMJtqGkl0wypG15gLbUP?usp=sharing

$g = 44$



blow-up of X | 2-secant flip | 3-secant flip

- Sequence of antiflips of (twisted) projective bundles

  $-K$ becomes "more ample"

- The sequence continues with antiflips over $X^{[4]}, \dots, X^{[10]}$ (Hilbert schemes of points on X)

- followed by an antiflip over $M^{(3)}_4$ ← rank of sheaves, ← dimension
  (HK moduli space of sheaves on K3 surface X),
- two simultaneous disjoint antiflips (over $X^{[11]}$ and $M^{(3)}_{10}$)
- The result is a small modification $\mathrm{Bl}_X \mathbb{P}^{44} \dashrightarrow M_{12}$,

(isomorphism in codimension 1)

M_{12} is Fano! $\Rightarrow$ $\mathrm{Bl}_X \mathbb{P}^{44}$ is log Fano

Class of log Fano varieties is closed under SQM and contractions (Prokhorov–Shokurov)

$$
\begin{tikzcd}
 & & \tilde{M} \arrow[lldd, "\tau_-"'] \arrow[rrdd, "\tau_+"] & & \\
 & & \mathbb{P}^- \times_B \mathbb{P}^+ \arrow[u, hook, "j"] \arrow[ld] \arrow[rd] & & \\
M^- \arrow[rrdd] & \mathbb{P}^- \arrow[l, hook'] \arrow[rd] & & \mathbb{P}^+ \arrow[r, hook, "i"] \arrow[ld, "\pi"'] & M^+ \arrow[lldd] \\
 & & B \arrow[d, hook, "j"] & & \\
 & & \bar{M} & &
\end{tikzcd}
$$

- Semi-orthogonal decompositions under (twisted) standard antiflips

$\mathrm{rk}\,\mathbb{P}^- < \mathrm{rk}\,\mathbb{P}^+$

Brauer class twist

- $D^b(M^+) = \langle D^b(M^-), D^b(B), D^b(B,\beta), \ldots, D^b(B,\beta^k) \rangle$

  $(\tau_+)_* \tau_-^*$ $\quad i_*\pi^*$ $\quad i_*(\pi^*(-)\otimes \mathcal{O}_\pi(1))$ $\quad i_*(\pi^*(-)\otimes \mathcal{O}_\pi(k))$

  (Bondal-Orlov) $\quad$ Belmans-Fu-Raedschelders (untwisted)

  Aravena (twisted case)

- $k+1 = \operatorname{rk} P^+ - \operatorname{rk} P^-$

Iterating over a sequence of antiflips $M_1 \dashrightarrow M_2 \dashrightarrow \ldots$, we get

- **Corollary** K3 surface X is a Fano visitor (if $g \not\equiv 3 \bmod 4$)
- **Bonus** There exist HK varieties of arbitrary dimension that are Fano visitors, e.g. $X^{[r]}$, $r \sim g/4$
- Where does the sequence of antiflips come from?

Remarks • Each M_i is a Lagrangian submanifold in the HK moduli space of Bridgeland stable objects on X

• If g is even (and possibly if g is odd), there exists a Poincaré family $\mathcal{E}$ on $X \times M_i$ and

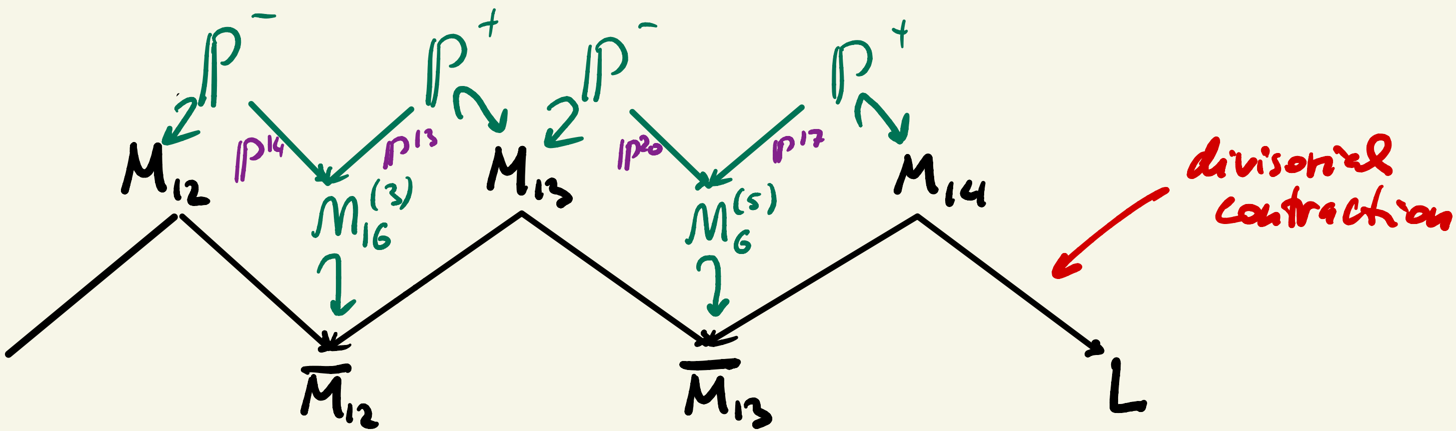
FM $\Phi_{\mathcal{E}}: D^b(X) \to D^b(M_i)$ is also f.f.!

• $\Phi_{\mathcal{E}}$ is a very different functor than the "torsion" functor defined by iterating standard antiflips but the proof "mutates" one into another

• I will explain the technology but for a different moduli space: of vector bundles on curves

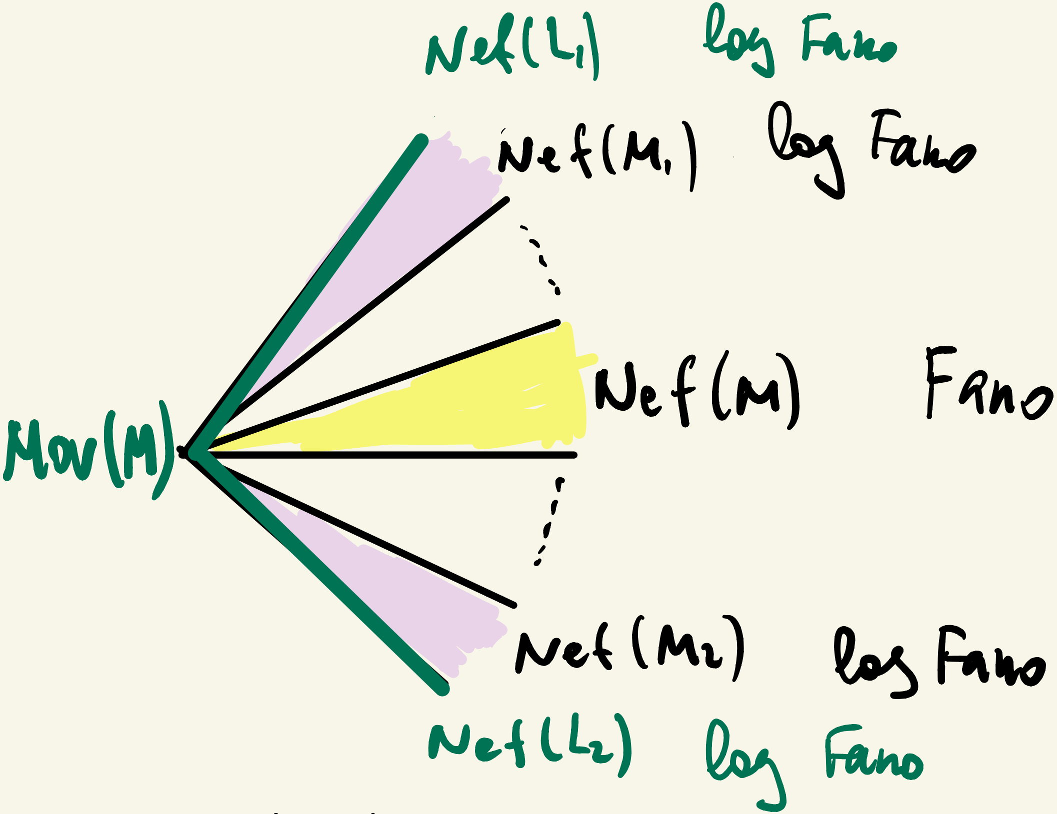
what happens to the S.O.D.'s
beyond the Fano model?

In our example of a $g=44$ K3 surface, we can continue beyond $\mathrm{Bl}_x\mathbb{P}^{44} \dashrightarrow M_{12}$:



In this direction, $D^b(-)$ decreases – but how?

More generally, consider a 2-ray game:



L

$\uparrow$ extremal contraction

M_1

$\uparrow$

small

M

modifications

$\downarrow$

M_2

$\downarrow$ extremal contraction

N

For simplicity, suppose all these varieties are smooth

Two-ray game conjecture: there exist S.O.Ds

$\langle A_1 \dots A_s \rangle = D^b(L_1)$

$\langle A_1, \dots A_s \; B_1 \dots B_t \rangle = D^b(M_1)$

compatibility with contractions

$\langle A_1 \dots A_s \; B_1 \dots B_t \; G \dots G \dots \rangle = D^b(M)$

Braid group action

$\langle A_1' \dots A_s' \; B_1' \dots B_t' \; G' \dots G' \dots \rangle = D^b(M)$

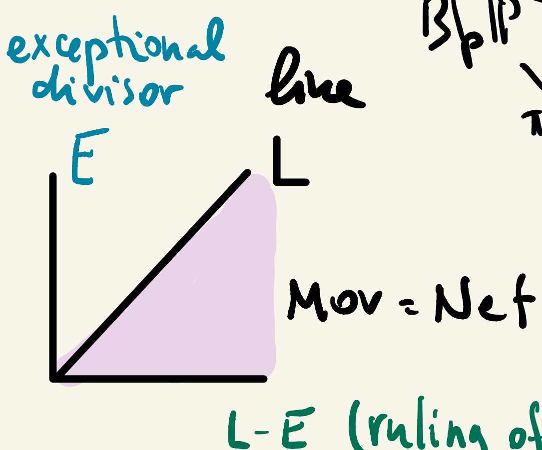
$\langle A_1', \dots A_s' \; B_1' \dots B_t' \rangle = D^b(M_2)$

$\langle A_1' \dots A_s' \rangle = D^b(L_2)$

compatibility with contractions

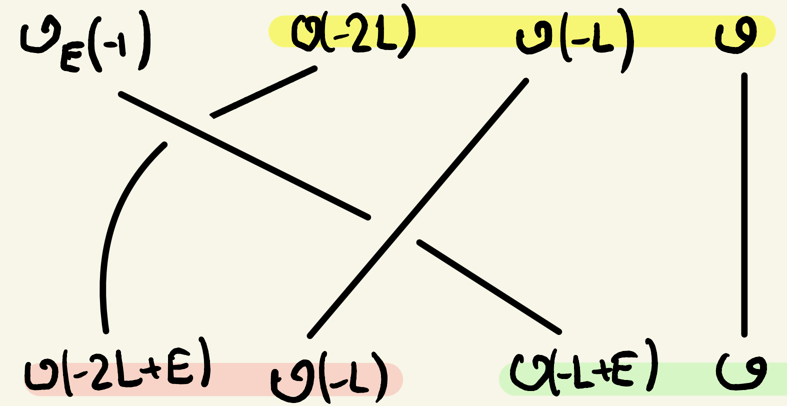
- Notion of compatibility depends on geometry of contractions
- For projective bundles, blow-ups, and standard flips compatibility is given by the Bondal-Orlov package

Example



$$\mathrm{Bl}_p\mathbb{P}^2 \to \mathbb{P}^2, \qquad \mathrm{Bl}_p\mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^1$$

$D^b(\mathbb{P}^2)$



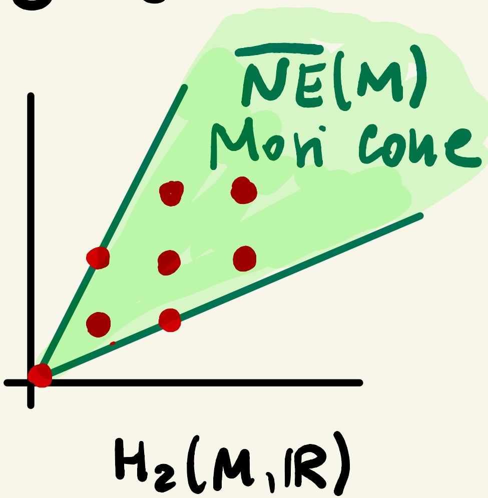
$\pi^* D^b(\mathbb{P}^1) \otimes \mathcal{O}(-L)$ $\qquad$ $\pi^* D^b(\mathbb{P}^1)$

$D^b(\mathbb{P}^1) = \langle \mathcal{O}(-1), \mathcal{O} \rangle$

- If this doesn't sound too unrealistic, here are even more outlandish conjectures about D^b(Fano):
- Kontsevich Conjecture $D^b(M)$ has a "canonical SOD" (unique up to mutation) indexed by eigenvalues in the quantum spectrum of M.
- Braiding Conjecture Mutation in two-ray games is induced by the monodromy of the quantum spectrum
- To state this conjecture rigorously, I need to introduce some notation

(standard for the Imperial College)

Full exposition in these notes, but only highlights in the lecture:



$B = \operatorname{Spec} k[\overline{NE}(M) \cap H_2(M,\mathbb{Z})]$

$d \in \overline{NE}(M) \cap H_2(M,\mathbb{Z}) \rightsquigarrow Q^d \in \mathcal{O}(B)$

(affine toric variety)

$0 \in B$ "large complex structure limit"

Small quantum cohomology:

$QH(M)$ is a sheaf of finite-dimensional algebras on B (associative, commutative, ...)

$QH(M)|_{0\in B} \cong H^*(M,\mathbb{C})$

- Convex polyhedral cones

$\mathrm{Nef}(M) \subset N^1(M) = H^2(M,\mathbb{R})$ (recall: M is Fano)

$NE(M) \subset N_1(M) = H_2(M,\mathbb{R})$

- $\Rightarrow$ affine toric variety B with cone $\mathrm{Nef}(M)$

$\mathcal{O}(B)$ is a semigroup algebra with basis

Q^d for $d \in NE(M) \cap H_2(M,\mathbb{Z})$ (Novikov variables)

- $$H^2(M,\mathbb{C}) \longrightarrow \underset{\text{open torus}}{H^2(M,\mathbb{C}^*)} \hookrightarrow B$$

$$\tau \longmapsto e^{\tau} \longmapsto * \qquad Q^d(*) = e^{\int_d \tau}$$

$$-\infty \longrightarrow 0 \in B$$

large complex structure limit

Small quantum cohomology $QH(M)$

- free $\mathcal{O}(B)$-module $\mathcal{O}(B)\otimes H^*(M,\mathbb{C})$
- algebra structure

$$\langle \alpha * \beta, \gamma \rangle = \sum_{d \in NE(M) \cap H_2(M,\mathbb{Z})} \langle \alpha, \beta, \gamma \rangle_d \, Q^d$$

(using Poincaré duality)

where $\langle \alpha, \beta, \gamma \rangle_d = \int_{\overline{M}_{0,3,d}^{vir}(M)} ev_1^*\alpha \cup ev_2^*\beta \cup ev_3^*\gamma$

moduli space of stable genus 0 maps $C \xrightarrow{f} M$ with 3 marked points and class $f[C] = d$

Example $QH(M)|_{0 \in B} \cong H^*(M,\mathbb{C})$

$\text{virdim}\, \overline{M}_{0,n,d}(M) = -K_M \cdot d + \dim M + n - 3$

$\Rightarrow \langle \alpha, \beta, \gamma \rangle_d = 0$ unless $\frac{1}{2}(\deg\alpha + \deg\beta + \deg\gamma) = -K_M \cdot d + \dim M$

$\Rightarrow$ • $\langle \alpha, \beta, \gamma \rangle_d = 0$ for all but finitely many d

• $QH(M)$ is a graded algebra

$wt(Q^d) = -K_X \cdot d$

$wt(\alpha) = \frac{1}{2} \deg \alpha$

• If α is represented by a Poincaré-dual cycle V

$\Rightarrow$ $wt\,[V] = \text{real codim}(V)/2$

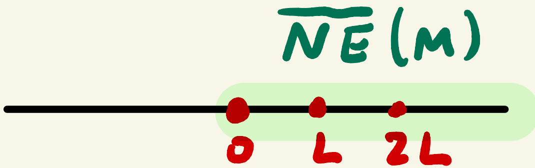
$= \text{complex codim}(V)$ (for complex cycles)

Example: $1 = [M]$ unity of $QH(M)$
(weight 0)

Def Quantum spectrum $QS(M) \subset B \times \mathbb{C}$ is the spectrum of the linear operator $-K_M \curvearrowright QH(M)$

Example: $QS(M)_{0 \in B} = \{0\}$ fat point
because $(-K_M)\cup\cdot$ is nilpotent

Example: $M = \mathbb{P}^2$



$$\begin{array}{cc} & \begin{array}{ccc} 1=[\mathbb{P}^2] & [L] & [pt] \\ \downarrow & \downarrow & \downarrow \end{array} \\ -K = 3L \curvearrowright & \begin{bmatrix} 0 & 0 & 3Q^L \\ 3 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \end{array}$$

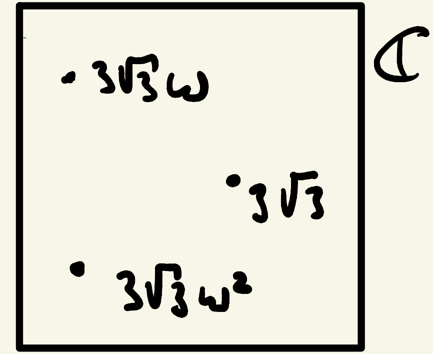
Basis $1 = [\mathbb{P}^2]$ $[L]$ $[pt]$

Characteristic polynomial $T^3 - 27 Q^L$

Homogeneity: $wt(T) = 1$, $wt(Q^L) = -K \cdot L = 3$

Dehomogenize: $Q^L = 1$

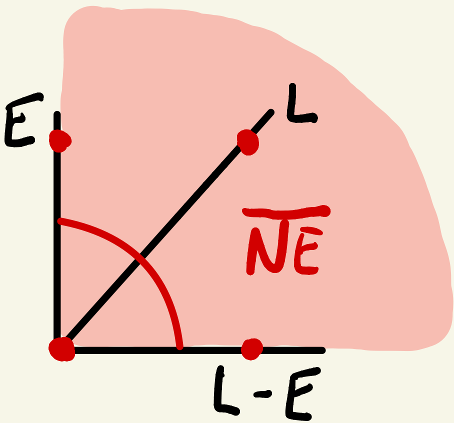
$QS(\mathbb{P}^2) =$



In general, $QS(M) \subset B \times \mathbb{C}$ is preserved by the $\mathbb{C}^*$-action with weights
$$\mathrm{wt}(Q^d) = -K \cdot d > 0$$
$$\mathrm{wt}(T) = 1$$

$\Rightarrow$ we can view $QS(M)$ as a subset in the total space of the line bundle $\mathcal{O}(1)$ on $B \setminus \{0\} / \mathbb{C}^* = \mathbb{P}^1$ if $b_2(M) = 2$: will assume for simplicity

Example: $M = Bl_p \mathbb{P}^2$



$(-K)\star -$	[M]	[L]	[E]	[pt]
[M]	0	$2q_2$	$2q_2$	$3q_1q_2$
[L]	3	0	0	$2q_2$
[E]	-1	0	$-q_1$	$-2q_2$
[pt]	0	3	1	0

$B = A^2$

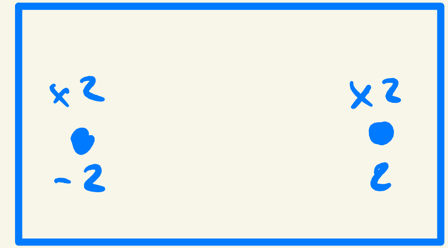
$q_1 = Q^E$

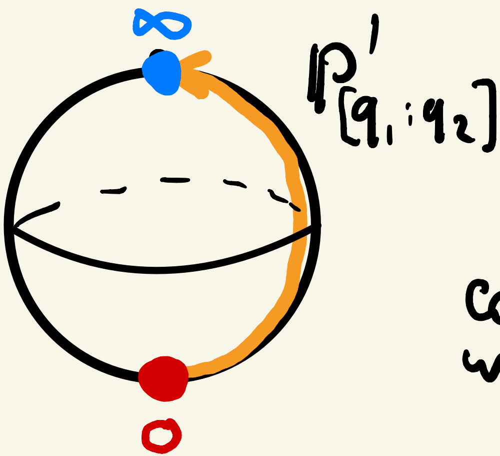
$q_2 = Q^{L-E}$

Char Poly $T^4 + q_1 T^3 - 8q_2 t^2 - 36 q_1 q_2 t - 27 q_1^2 q_2 + 16 q_2^2$

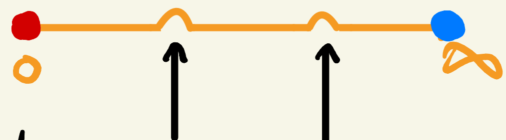
Monodromy of eigenvalues

$q_1 = 0 \quad q_2 = 1 \qquad T^4 - 8t^2 + 16 = (T-2)^2(T+2)^2$



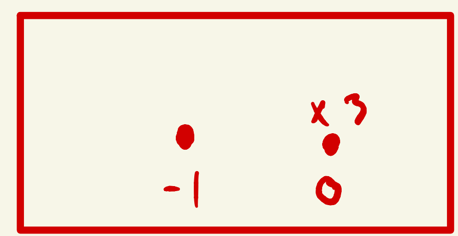


Contour:

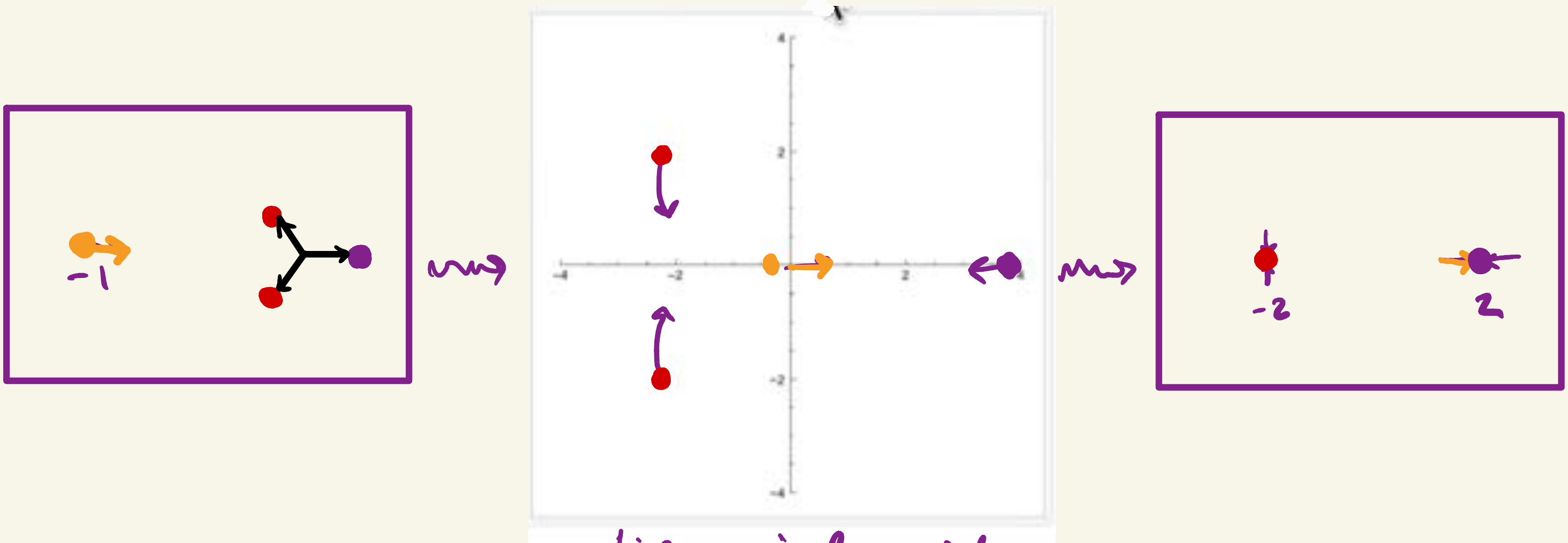


contour avoids points where CharPoly has multiple roots

monodromy of eigenvalues



$q_1 = 1 \quad q_2 = 0 \qquad T^4 + T^3 = (T+1)T^3$



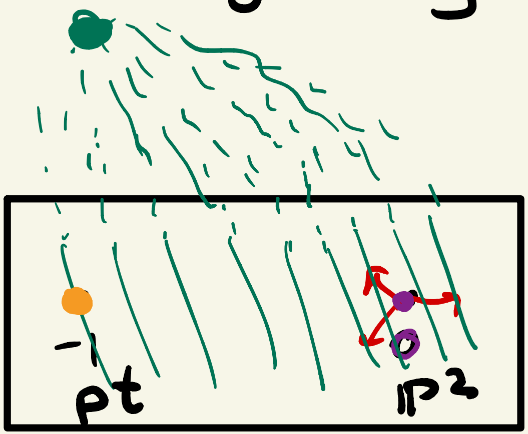
$q_2 = 0$

anticanonical point
$(q_1 = q_2 = 1)$

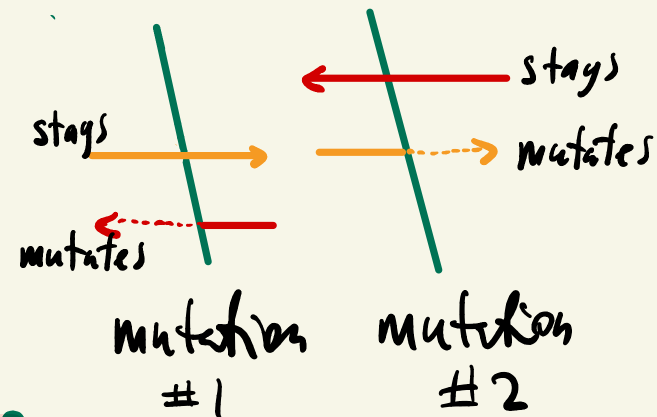
$q_1 = 0$

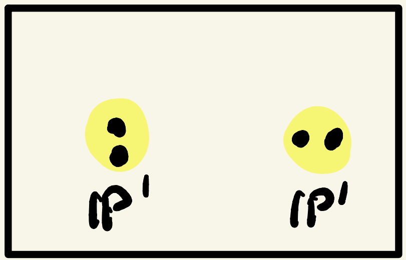
Watch movies at https://people.math.umass.edu/~tevelev/eigenvalues.html

- To compute the mutation, choose "grading" by connecting eigenvalues to a reference point
- The grading determines the order of the SOD



$\mathcal{O}_E(-1)$ $\mathcal{O}(-2L)$ $\mathcal{O}(-L)$ $\mathcal{O}$

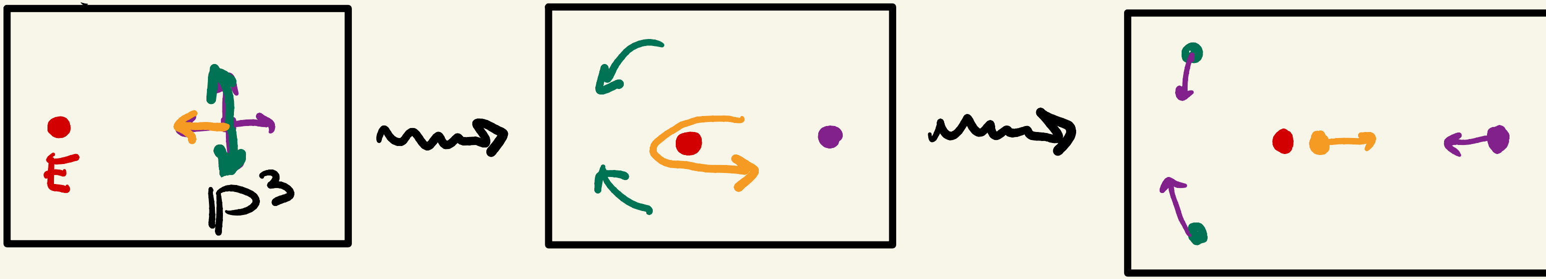




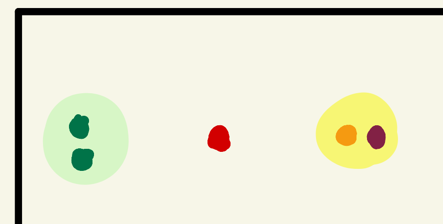
$D^b(\mathbb{P}^1)\otimes\mathcal{O}(-L)$ $D^b(\mathbb{P}^1)$

- This shows how "compatibility" with projective bundles look like. How about more complicated contractions?

Example $Bl_E(\mathbb{P}^3)$ $E = Q_1 \cap Q_2$ degree 4 genus 1



anticanonical point



$D^b(\mathbb{P}^1)\otimes \mathcal{O}(-H)$ $D^b(\mathbb{P}^1)$

The second contraction is a quadric fibration $Bl_E \mathbb{P}^3 \xrightarrow{[Q_1:Q_2]} \mathbb{P}^1$

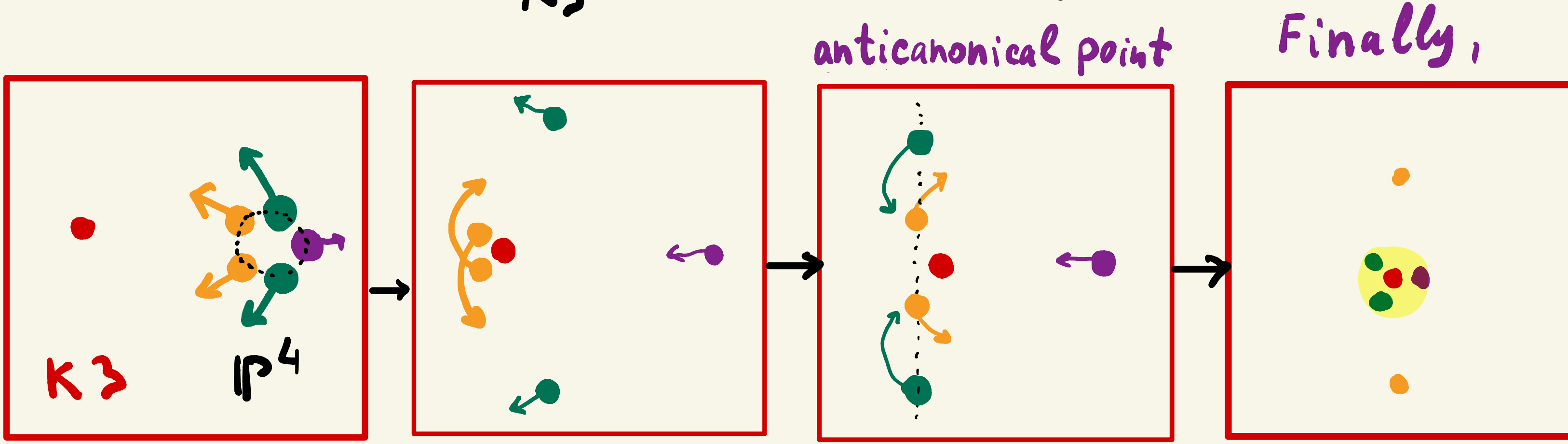
Theorem (Kuznetsov) D^b(Quadric fibration)

$= \langle D^b$(sheaves of even clifford algebras on the base), $\underbrace{D^b(\text{base}), \dots, D^b(\text{base})}\rangle$

like for the projective bundle

How about singular contractions?

Example $M = \mathrm{Bl}_{K3}\mathbb{P}^4$, $K3 = Q_2 \cap Q_3$



The second contraction $M \longrightarrow X_4 \subset \mathbb{P}^5$ singular cubic fourfold

$D^b(X_4) = \langle \mathcal{A}, \mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O} \rangle$ Kuznetsov component

Theorem (Kuznetsov) $\mathcal{A}$ has NCR $= D^b(K3)$

- Justification: Kontsevich's Homological Mirror Symmetry

$D^b(M) \simeq FS(M^\vee, w)$, where $w: M^\vee \to \mathbb{C}$ is a function (superpotential)

Fukaya-Seidel

- Mirror Symmetry switches Kähler & complex moduli

$\Rightarrow$ variation over B gives a family of mirrors

- Example: $M = Bl_p\, \mathbb{P}^2 \Rightarrow M^\vee = (\mathbb{C}^*)^2$, $W = x + y + \frac{q_1 q_2}{xy} + \frac{q_2}{x}$
- $QS(M) = \{\text{critical values of } W\}$

- $FS(M^\vee, w)$, by definition, is constructed from blocks indexed by $QS(M)$ glued into an SOD using a line field ("grading")
- Critical locus of the superpotential over $\bullet \in QS(M)$ should be a mirror of Block $\subset D^b(M)$
- Examples:

$Bl_E\, \mathbb{P}^3 \rightsquigarrow$ fiber over $\bullet$ is an elliptic curve

$Bl_{K3}\, \mathbb{P}^4 \rightsquigarrow$ fiber over $\bullet$ is a K3 surface

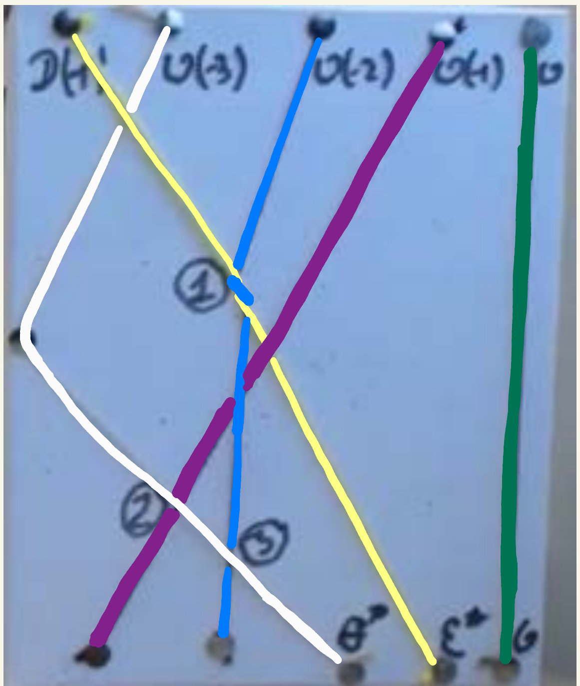
An example where both contractions are birational onto smooth Fano manifolds:

Example $M = Bl_C \mathbb{P}^3 \xrightarrow{\text{1st contraction}} \mathbb{P}^3 \overset{\deg 5}{\hookleftarrow} C \quad g=2$

MM 2-19 $\quad$ 211

$D^b(C)$ $\quad Bl_{\mathbb{P}^1} Q_1 \cap Q_2 \xrightarrow{\text{2d contraction}} Q_1 \cap Q_2 \overset{\text{line}}{\hookleftarrow} \mathbb{P}^1$

$Q_1 \cap Q_2 \subset \mathbb{P}^5$



Eigenvalue mutation proves a theorem of Bondal and Orlov

$$D^b(Q_1 \cap Q_2) = \langle \underset{\mathcal{O}(-1)}{D^b(pt)}, \underset{\Phi}{D^b(C)}, \underset{\mathcal{O}}{D^b(pt)} \rangle$$

$\Phi : D^b(C) \longrightarrow D^b(Q_1 \cap Q_2)$ (Newstead)

$\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{SU_C(2,\Lambda)}$ genus $=2$

fixed odd det

$\mathcal{E}$ Poincaré vector bundle on $C \times SU_C(2,\Lambda)$, $\Phi = \Phi_{\mathcal{E}^*}$

To generalize this to every genus, we use secant flips of Bertram–Thaddeus. Choose $\Lambda \in \mathrm{Pic}^d C$, $d = 2g-1$

divisorial contraction | standard antiflips | divisorial contraction

$$\begin{tikzcd}[column sep=small]
E \arrow[r, hook] \arrow[d] & \mathrm{Bl}_C \mathbb{P}^{3g-3} = M_1 \arrow[dl] \arrow[dr] & & M_2 \arrow[dl] \arrow[dr] & & M_3 \arrow[dl] & \cdots\cdots & & M_{g-1} \arrow[dl] \arrow[dr] & & Z \arrow[ll, hook'] \arrow[d] \\
C \arrow[r, hook] & \mathbb{P}^{3g-3} = M_0 & & \overline{M}_1 & & \overline{M}_2 & \cdots\cdots & \overline{M}_{g-2} & & SU_C(2,\Lambda) & B \arrow[l, hook'] \\
 & |\omega_C \otimes \Lambda| & & \cup & & \cup & & \cup & & & \\
 & & & \mathrm{Sym}^2 C & & \mathrm{Sym}^3 C & \cdots\cdots & \mathrm{Sym}^{g-1} C & & &
\end{tikzcd}$$

$M = M_{g-1}$ is Fano (in particular, $\mathrm{Bl}_C \mathbb{P}^{3g-3}$ is log Fano)

$$D^b(M) = \langle \underbrace{pt \ \dots \ pt}_{3g-2} \ \underbrace{C \ \dots \ C}_{3g-5} \ \underbrace{\mathrm{Sym}^2 C \dots \mathrm{Sym}^2 C}_{3g-8} \ \dots \ \underbrace{\mathrm{Sym}^{g-1} C}_{1} \rangle$$

by the Bondal-Orlov package.

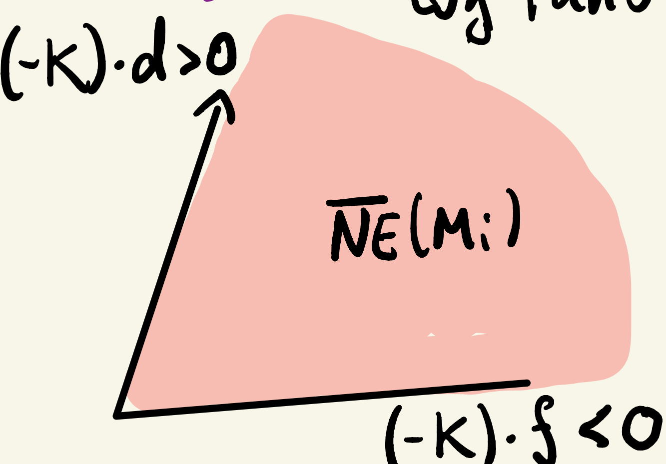
$\Rightarrow$ C is a Fano visitor

Bonus $\mathrm{Sym}^k C$ is a Fano visitor for $k \leq g-1$ (choosing $d \gg 2g-1$, one can embed every $\mathrm{Pic}^k C$, and therefore $\mathrm{Jac}\, C$)

Small quantum cohomology $QH(M)$ is not known but we can speculate how the quantum spectrum **$QS(M)$** looks like. What happens under flips?

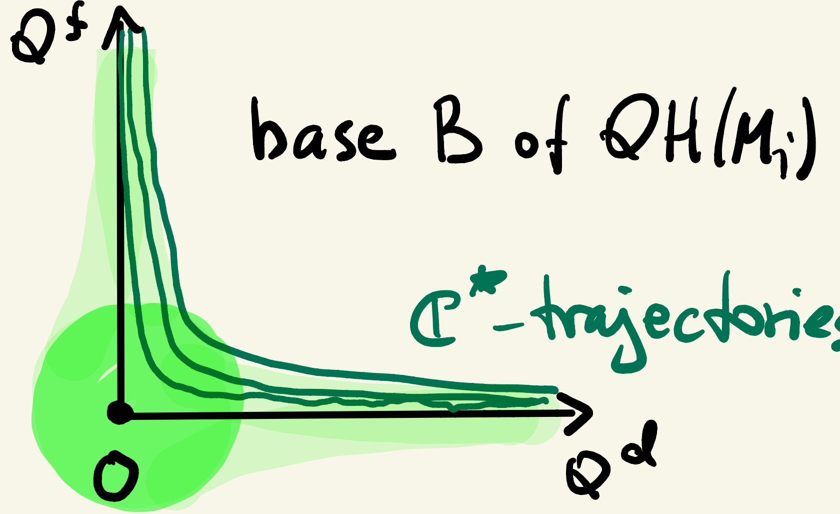
Curve d was antiflipped (or extracted) before

log Fano M_i:



Curve f needs to be antiflipped

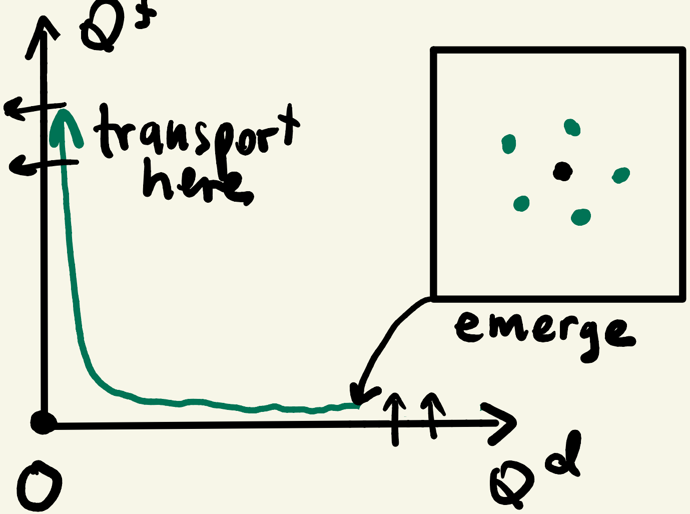
base B of $QH(M_i)$



Conjecture: QH converges in some neighborhood of $0 \in B$

$\Rightarrow$ converges near the walls using $\mathbb{C}^*$-trajectories.

$\Rightarrow$ Emerged quantum spectrum is transported along trajectories



Theorem (Shen–Shoemaker)

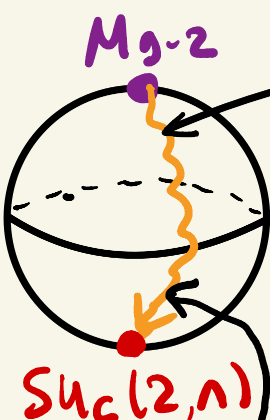
Quantum spectrum of the standard antiflip

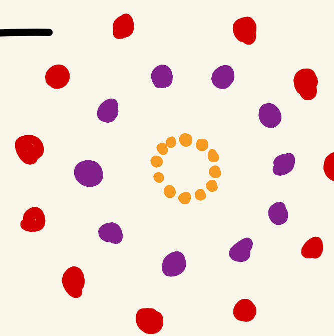
$$\mathbb{P}^s \searrow \overline{M}_i \swarrow \mathbb{P}^r$$

$\pm\{$roots of unity of order $r-s\}$

$\pm$ negative if s is even

$\Rightarrow$ After a sequence of standard antiflips, $QS(M)$ Fano will look like concentric circles



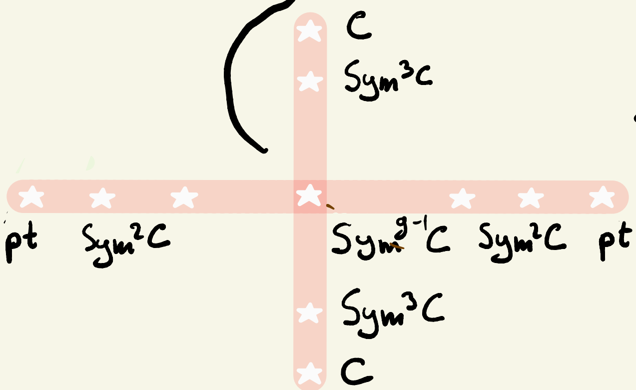


M_{3g-2}, $-M_{3g-5}$, M_{3g-8}, …

pt, C, $\mathrm{Sym}^2 C$

emerging from the origin

Now follow the contour…



Close to the south pole, we should see the emerging quantum spectrum of $SU_C(2,\Lambda)$, which was computed by del Baño

So, eventually, this quantum magic will prove

Theorem (T.-Torres)

$D^b(SU_C(2,\Lambda))$ has SOD with blocks $D^b(Sym^k C)$
2 blocks for $k < g-1$, 1 block for $k = g-1$

Remark This includes Narasimhan's theorem, mentioned earlier, that $\Phi_{\mathcal{E}} : D^b(C) \hookrightarrow D^b(SU_C(2,\Lambda))$ is fully faithful

The technology is not there yet, so I will prove SOD differently, by "guessing" and analyzing the mutation.