Semi-orthogonal decompositions of Fano & moduli

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classical geometry

X smooth proj. variety $\rightsquigarrow$

non-commutative geometry

category of sheaves on X (and similar categories)

$D^b(X)$ objects = bounded complexes of coherent sheaves

$\overset{\cong}{D^b(X)}$ — " — " ——— " — vector bundles

Example

F/X vector bundle of rank r

generalized complete intersection

$Y \subset X$ zero locus of $s \in H^0(X, F)$ $\operatorname{codim}_X Y = r$

$\Rightarrow \mathcal{O}_Y \underset{D^b(X)}{\cong} [\Lambda^r F^* \to \ldots \to F^* \xrightarrow{s} \mathcal{O}_X]$ Koszul complex

- $D^b(X)$ is not an abelian but a triangulated category:

morphism $A \xrightarrow{f} B \rightsquigarrow$ exact triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$

via the mapping cone construction

shift functor in $D^b[X]$

- $\mathrm{Coh}\,X \hookrightarrow D^b(X)$ Fully-faithful (preserves Hom spaces)

$A \mapsto [\cdots \to 0 \to \underset{\text{degree } 0}{A} \to 0 \to \cdots]$

- $0 \to A \to B \to C \to 0 \longmapsto A \to B \to C \to A[1] \to$

short exact sequence ... exact triangle

take exact triangles to exact triangles & commute with shift

$$\mathrm{Ext}^k(A,B) \cong \mathrm{Hom}(A, B[k])$$

(use as definition if $A, B \in D^b(X)$)

- Functors of triangulated categories are presumed to be exact
- Classical functors of abelian categories are (often) not exact but their derived functors are exact

Focus of Talks: fully-faithful functors $\Phi : D^b(X) \to D^b(Y)$
(including equivalences)

another smooth projective variety

- Fully-faithful functors are **rare**

Non-example $\{p\} \overset{i}{\hookrightarrow} X$

$$i_* : D^b(pt) = D^b(\mathrm{Vect}_k) \hookrightarrow D^b(X)$$

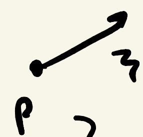
$$k \longmapsto k(p)$$ skyscraper sheaf

$\mathrm{Hom}(k,k) = k$ and $\mathrm{Hom}(k(p),k(p)) = k$

$\mathrm{Hom}(k,k[1]) = 0$ but $\mathrm{Hom}(k(p),k(p)[1]) = \mathrm{Ext}^1(k(p),k(p)) = T_{X,p}$

$\Rightarrow$ Not fully faithful!

Tangent vector



$$\mathcal{O}_X \longrightarrow \mathcal{O}_Z$$
$$f \mapsto (f(p), f'_z(p))$$

extension $0 \to k(p) \to \mathcal{O}_Z \to k(p) \to 0$

Upshot: the point is **rigid** on its own but **moves** in X

Example Fully faithful $D^b(p) \longrightarrow D^b(Y)$

$$k \longmapsto E \qquad \mathrm{Ext}^i(E,E) = \begin{cases} \mathbb{C} & i=0 \\ 0 & i \neq 0 \end{cases}$$

E is called an "exceptional" vector bundle (or sheaf or complex)

Example Y Fano $\Rightarrow H^i(Y, \mathcal{O}_Y) = 0,\ i>0 \Rightarrow \mathcal{O}_Y$ exceptional

$$\mathcal{O}_Y = K_Y + \underbrace{(-K_Y)}_{\text{ample}} \quad \text{Kodaira vanishing}$$

More generally, if $\Phi : D^b(X) \to D^b(Y)$ is f.f. $\Rightarrow$

- $\forall p \in X$,

$$\mathrm{Ext}^i(\Phi(k(p)), \Phi(k(p))) = \mathrm{Ext}^i(k(p), k(p)) = \begin{cases} \mathbb{C} & i=0 \\ \Lambda^i T_p X & 0 < i \leq \dim X \\ 0 & \text{if } i<0 \text{ or } i > \dim X \end{cases}$$

- $\forall p \neq q$

$$\mathrm{Ext}^i(\Phi(k(p)), \Phi(k(q))) = \mathrm{Ext}^i(k(p), k(q)) = 0$$

Bondal-Orlov criterion Φ is f.f. $\Leftrightarrow$ green conditions observation If Φ is f.f. then X is a moduli space of objects in $D^b(Y)$

This has two ingredients:

(1) (Orlov) Φ f.f $\Rightarrow \exists\, \mathcal{E} \in D^b(X \times Y)$ (Fourier-Mukai kernel)

such that $\Phi = \Phi_{\mathcal{E}} = \pi_{Y*}(\pi_X^*(-) \otimes \mathcal{E})$ (Fourier-Mukai transform)

$\Rightarrow$ Function $p \in X \longmapsto \mathcal{E}_p := \Phi(k(p)) \in D^b(Y)$ can be upgraded to

$\{$T-points of $X\} \longmapsto \{$families of objects in $D^b(Y)$ parametrized by T$\}$ via pullback of $\mathcal{E}$

(2) f.f. property $\Rightarrow X$ is a complete moduli space of objects in $D^b(Y)$:

- $\mathrm{Ext}^k(\mathcal{E}_p, \mathcal{E}_p) = 0 \quad k < 0$ (needed to define moduli of objects in $D^b(Y)$)
- $\mathrm{Hom}(\mathcal{E}_p, \mathcal{E}_p) = k \Rightarrow$ simple object
- $\mathrm{Ext}^1(\mathcal{E}_p, \mathcal{E}_p) = T_pX$ infinitesimal deformations are controlled by X
- obstruction: $\mathrm{Ext}^1(\mathcal{E}_p, \mathcal{E}_p) \to \mathrm{Ext}^2(\mathcal{E}_p, \mathcal{E}_p) = \Lambda^2 T_pX$ vanishes (it is symmetric)
- $\mathrm{Hom}(\mathcal{E}_p, \mathcal{E}_q) = 0$ if $p \neq q \Rightarrow \mathcal{E}_p$ moves with p (not by some other reason)

One can also ask: are $E_p \in D^b(Y)$ stable with respect to some notion of stability on Y?

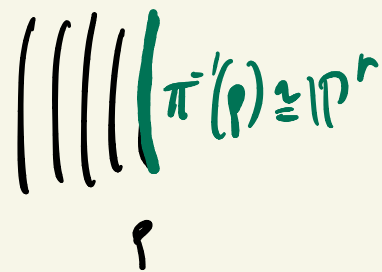
Example (Drezet-Le Potier, 1985) Introduced and constructed exceptional vector bundles on $\mathbb{P}^2$, proved their stability

Examples of f.f. functors:

- $Y \xrightarrow{\pi} X$ projective bundle

  $\pi^* : D^b(X) \to D^b(Y)$ f.f.

  $k(p) \mapsto \mathcal{O}_{\pi^{-1}(p)}$



- $Y = Bl_Z X$ blow-up

$$\begin{tikzcd} E \arrow[r, hook, "i"] \arrow[d, "\pi"'] & Y \arrow[d] \\ Z \arrow[r, hook] & X \end{tikzcd}$$

$i_* \circ \pi^* : D^b(Z) \to D^b(Y)$ f.f.

$k(p) \to \mathcal{O}_{\pi^{-1}(p)}$

- $Y \xrightarrow{f} X$ $\quad Rf_* \mathcal{O}_Y = \mathcal{O}_X$

(e.g. blow-up or projective bundle) $\Rightarrow$ $Lf^* : D^b(X) \to D^b(Y)$ f.f.

More examples: Extremal contractions

cone generated by effective curves in Y

Mori Cone $\overline{NE}(Y) \subset N_1(Y) \subset H_2(Y, \mathbb{R})$

Nef Cone $\mathrm{Nef}(Y) \subset N^1(Y) \subset H^2(Y, \mathbb{R})$

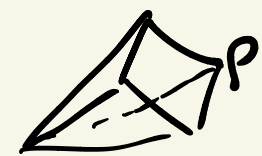
convex dual of $\overline{NE}(Y)$

Neron-Severi subspaces spanned by curves/divisors

Cone + BPF Theorem • $\overline{NE}(Y) \cap (-K)^{>0}$ locally polyhedral cone

- generators $\rho = [\mathbb{P}^1]$ $\quad \mathbb{P}^1 \subset Y$ rational curve
- $\exists \ f : Y \to X$, $\ f(C) = pt \Leftrightarrow [C] \in \rho$
- $Rf_* \mathcal{O}_Y = \mathcal{O}_X \Rightarrow Lf^* : D^b(X) \hookrightarrow D^b(Y)$ fully-faithful

if X is smooth. Unfortunately, X is often singular



Adjoint functors

(Mukai) Any FM functor $\Phi_{\mathcal{E}}: D^b(X) \to D^b(Y)$ has adjoint functors

$\Phi^* = \Phi_{\mathcal{E}^\vee} \circ S_Y$ and $\Phi^! = S_X \circ \Phi_{\mathcal{E}^\vee}$, where

(left adjoint) $\mathcal{E}^\vee = R\mathcal{H}om(\mathcal{E}, \mathcal{O}_{X\times Y})$ (right adjoint)

$S_Y: D^b(Y) \to D^b(Y)$ is the Serre functor $T \mapsto T \otimes \omega_Y[\dim Y]$

$\Rightarrow$ If $\Phi: D^b(X) \to D^b(Y)$ is f.f. then $\mathcal{A} = \Phi(D^b(X))$ is an admissible subcategory of $D^b(Y)$

full triangulated subcategory with adjoint functors for inclusion

$\Rightarrow$ $\mathcal{B} = {}^{\perp}\mathcal{A} = \{B : \mathrm{Hom}(B, A) = 0 \;\; \forall A \in \mathcal{A}\}$ also admissible

$\Rightarrow$ $D^b(Y) = \langle \mathcal{A}, \mathcal{B} \rangle$ is a S.O.D.:

(1) $\mathcal{A}, \mathcal{B}$ full triangulated subcategories

(2) $\mathrm{Hom}(B, A) = 0 \quad \forall A \in \mathcal{A},\; B \in \mathcal{B}$

(3) $\forall\, T \in D^b(Y)$ $\exists$ exact triangle "projector functor"

$B \to T \to A \to B[1]$ here $A = \varphi^*(T)$

$\varphi : \mathcal{A} \hookrightarrow D^b(Y)$

Likewise, we have a SOD $D^b(Y) = \langle \mathcal{C}, \mathcal{A} \rangle$,

where $\mathcal{C} = \mathcal{A}^{\perp}$. Furthermore, $\mathcal{C} \cong \mathcal{B}$ as triangulated categories

In fact, $\mathcal{C} = \mathcal{B} \otimes \omega_Y$ (use Serre duality)

This is an example of a mutation $\langle \mathcal{A}, \mathcal{B} \rangle$

$\langle \mathcal{C}, \mathcal{A} \rangle$

Longer SOD $D^b(Y) = \langle \mathcal{A}_1 \ldots \mathcal{A}_n \rangle$

- Filtration $0 = \mathcal{T}_0 \subset \ldots \subset \mathcal{T}_n = D^b(Y)$ by admissible subcategories.
- $\mathcal{T}_i = \langle \mathcal{T}_{i-1}, \mathcal{A}_i \rangle$ 2-step SOD

Example Y index r Fano $\omega_Y^{-1} \cong \mathcal{O}(r)$

$\Rightarrow D^b(Y) = \langle \mathcal{A}, \mathcal{O}(-r+1), \ldots, \mathcal{O} \rangle$

$\uparrow$ **Kuznetsov component**

Example $D^b(\mathbb{P}^n) = \langle \mathcal{O}(-n), \ldots, \mathcal{O} \rangle$ **Beilinson exc. collection**

(the only case when the Kuznetsov component $= 0$)

Relative version (Orlov) $X = \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} Y$ projective bundle of rank n

$\Rightarrow D^b(X) = \langle \pi^* D^b(Y) \otimes \mathcal{O}_\pi(-n), \ldots, \pi^* D^b(Y) \rangle$

Orlov blow-up Theorem

$$\begin{array}{ccc} X = Bl_Z Y & \xrightarrow{f} & Y \\ \cup & & \cup \\ E & \xrightarrow{\pi} & Z \end{array}$$

$D^b(X) = \langle \pi^* D^b(Z) \otimes \mathcal{O}_\pi(-k), f^* D^b(Y) \rangle \quad 1 \leq k \leq \operatorname{codim}(Z) - 1$

Braid group $\mathbb{B}_n$ acts on n-step SOD via generators:

$$\langle \mathcal{A}_1 \cdots\cdots \mathcal{A}_i \quad \mathcal{A}_{i+1} \cdots\cdots \mathcal{A}_r \rangle \quad \text{and}$$

$$\langle \mathcal{A}_1 \cdots L_{\mathcal{A}_i}(\mathcal{A}_{i+1}) \quad \mathcal{A}_i \cdots\cdots \mathcal{A}_r \rangle$$

$\cong\uparrow \mathbb{L}$ mutation equivalence functor

$\mathcal{A}_{i+1}$

Concretely, $T \in \mathcal{A}_{i+1} \Rightarrow \exists$ unique exact $\varphi\varphi^!(T) \to T \to \mathbb{L}(T)$

$$\varphi\varphi^!(T) \in \mathcal{A}_i, \quad T \in \mathcal{A}_{i+1}, \quad \mathbb{L}(T) \in L_{\mathcal{A}_i}(\mathcal{A}_{i+1})$$

$$\varphi : \mathcal{A}_i \hookrightarrow D^b(Y)$$

- Example: $\mathcal{A}_1 \ \mathcal{A}_2 \cdots \mathcal{A}_r$

$$\mathcal{A}\otimes\omega_X \ \ \mathcal{A}_1 \ \ \mathcal{A}_2 \ \ \mathcal{A}_{r-1}$$

which $D^b(Y)$ admit non-trivial SOD?

Example Suppose $D^b(Y) = \langle \mathcal{A}, \mathcal{B} \rangle$ and $\omega_Y = \mathcal{O}_Y$

Then $\langle \mathcal{A}, \mathcal{B} \rangle = \langle \mathcal{B} \otimes \omega_Y, \mathcal{A} \rangle = \langle \mathcal{B}, \mathcal{A} \rangle \Rightarrow D^b(Y) = \mathcal{A} \oplus \mathcal{B}$

$\forall p \in Y$, $\underset{\substack{\cap \\ \mathcal{A}}}{k(p)} \cong \underset{\substack{\cap \\ \mathcal{A}}}{A} \oplus \underset{\substack{\cap \\ \mathcal{B}}}{B}$ $\quad \mathrm{Hom}(k(p), k(p)) = k \Rightarrow k(p) \in \mathcal{A}$ or $k(p) \in \mathcal{B}$

$\Rightarrow Y$ is disconnected

This simple argument can be much improved.

Conjecture ω_Y nef & effective, Y connected $\Rightarrow D^b(Y)$ indecomposable

Th (Lin, based on work of Kawatani-Okawa)

$\bigcap_{L \in \mathrm{Pic}^0(Y)} \mathrm{Bs}\,|\omega_Y \otimes L|$ finite $\Rightarrow D^b(Y)$ indecomposable

(topologically trivial line bundles)

(Kawatani-Okawa proved the same statement for $Bs|\omega_Y|$)

Example $\quad C$ smooth projective curve

rank 1 stable pair

$$\mathrm{Sym}^k C = C \times \dots \times C / S_k$$
$$= \{\text{effective divisors of degree } k\}$$
$$= \{ (s, L) : L \in \mathrm{Pic}^k C,\ \mathcal{O}_C \xrightarrow{s} L,\ s \neq 0 \}$$

Th (Biswas-Gomez-Lee)

$$p_1 + \dots + p_k \in Bs|K| \iff h^0(p_1 + \dots + p_k) > 1$$

This shows that

$$k < \mathrm{gon}(C) \underset{\text{KO}}{\Longrightarrow} D^b(\mathrm{Sym}^k C) \text{ indecomposable}$$

But one can do better:

$$k < g \underset{\text{Lin}}{\Longrightarrow} D^b(\mathrm{Sym}^k C) \text{ indecomposable}$$

Abel-Jacobi map $\mathrm{Sym}^g C \xrightarrow{\Sigma} \mathrm{Pic}^g C$ is birational

$\Rightarrow D^b(\mathrm{Sym}^g C) = \langle\ ?\ , D^b(\mathrm{Pic}^g C)\rangle$

via $L\Sigma^*$

$\simeq D^b(\mathrm{Sym}^{g-2} C)$
(Toda)

Note that Σ is not an isomorphism over the Brill-Noether locus $B = \{L \in \mathrm{Pic}^g C : h^0(L) \geq 2\}$
$= \{L \in \mathrm{Pic}^g C : h^0(K-L) \geq 1\}$

A birational map $\mathrm{Sym}^{g-2} C \longrightarrow B$
$D \longmapsto \mathcal{O}(K-D)$
is a resolution of singularities

Toda constructed SOD of $D^b(\mathrm{Sym}^k C)$ for all k

$k \gg 0 \Rightarrow \mathrm{Sym}^k C \to \mathrm{Pic}^k C$ is a projective bundle

$\Rightarrow D^b(\mathrm{Sym}^k C) = \langle D^b(\mathrm{Pic}^k C), \dots, D^b(\mathrm{Pic}^k C)\rangle$ by Orlov theorem

Fano visitor Conjecture (Bondal) $\forall$ smooth projective variety X there exists a fully faithful $\Phi : D^b(X) \hookrightarrow D^b(Y)$, **$Y$ Fano**

This looks like saying that every X can be embedded in $\mathbb{P}^n$ but analogy is superficial: we can deform subvarieties of $\mathbb{P}^n$ but f.f. functors Φ do not deform since X is the moduli space of objects $\mathcal{E}_p \in D^b(Y)$ (except by applying automorphisms of X)

In fact, admissible subcategories $\mathcal{A} \hookrightarrow D^b(Y)$ also don't deform (and **deform uniquely** if we deform $Y \rightsquigarrow Y'$).

Caveat: If $\mathcal{A} = D^b(X)$ then deformed $\mathcal{A}' \subset D^b(Y')$ can be **non-geometric** (i.e $\neq D^b(X')$) Admissible subcategories $\mathcal{A} \subset D^b(Y)$ are examples of proper & smooth **non-commutative algebraic varieties**

Example: $Y \subset \mathbb{P}^5$ cubic fourfold $D^b(Y) = \langle \underset{\substack{\uparrow \\ D^b(K3) \text{ or deformation of } D^b(K3)}}{A}, \mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O} \rangle$.

Remark N.c. deformations are rare:
If $H^2(X, \mathcal{O}_X) = H^0(X, \Lambda^2 T_X) = 0 \Rightarrow \underbrace{HH^2(X)}_{\text{Hochschild cohomology}} = H^1(X, T_X)$ and all deformations are geometric

Fano visitor conjecture implies that Fano varieties are abundant, there are as many of them as smooth projective varieties.

Where to find Fano hosts?
Fano complete intersections (including zeros of sections of vector bundles) on key varieties like $\mathbb{P}^n$, flag varieties, Fano toric varieties, etc. are not enough even

to embed derived categories of curves $D^b(C)$ $g \gg 0$

(moduli of complete intersections are unirational

but M_g for $g \gg 0$ is not)

Remark Works for some varieties with unirational moduli,

Example • C hyperelliptic $\Rightarrow$ $D^b(C) \hookrightarrow D^b(Q_1 \cap Q_2)$ $\subset \mathbb{P}^{2g+1}$

(Bondal-Orlov)

• Any smooth complete intersection in $\mathbb{P}^n$ is a Fano visitor

(Kiem-Kim-Lee-Lee)

Let's brainstorm some ideas where else

to find Fano hosts

Idea 1 Construct Y as a moduli space of objects on X

Example: $X = pt \Rightarrow Y = pt$. So can't get complicated exceptional objects $\langle E \rangle \subset D^b(Y)$ but good enough!

Using universal families to construct fully-faithful functors is **classic**:

Example $X =$ abelian variety
$Y = Pic^0(X)$ dual abelian variety
$\mathcal{E}$ Poincaré line bundle on $X \times Pic^0 X$

Th (Mukai) $\Phi_{\mathcal{E}} : D^b(X) \to D^b(Pic^0 X)$ is an equivalence

Sketch Since both are indecomposable, it is enough to check $\Phi_{\mathcal{E}^\vee} : D^b(Pic^0 X) \to D^b(X)$ is fully-faithful.

By Bondal-Orlov, this is equivalent to

$L_1 \not\cong L_2 \in \mathrm{Pic}^0 X \Rightarrow \mathrm{Ext}^k(L_1, L_2) = H^k(X, L_1^* \otimes L_2) = 0 \quad \forall k$

Good exercise! If can't solve, resort to mirror symmetry

This calculation shows that checking the Bondal-Orlov criterion involves proving a lot of non-trivial vanishing theorems

It also shows that we have an obstruction:

Obstruction In general, $\mathrm{Pic}^0(X) \circlearrowright$ any moduli space Y on X $\otimes$

Action is typically not trivial (e.g. for moduli of vector bundles)

but Abelian varieties can't act on Fano varieties

$(Y \underset{|-mK|}{\hookrightarrow} \mathbb{P}^N \Rightarrow \mathrm{Aut}(Y) \hookrightarrow PGL_{N+1}$ linear algebraic group)

Solution Consider moduli of objects with **fixed determinant**

- **Example** C smooth projective curve of genus $g \geq 2$
- $SU_C(2,\Lambda)$ moduli space of semi-stable rank 2 vector bundles F on C with $\det F = \Lambda$
- Fano variety, $\dim = 3g-3$, $\mathrm{Pic} = \mathbb{Z} = \langle \Theta \rangle$
- $SU_C(2,\Lambda) \cong SU_C(2, \Lambda \otimes L^{\otimes 2}) \Rightarrow$ 2 cases only

  $F \longleftrightarrow F \otimes L$
- $\deg \Lambda$ odd

  $SU_C(2,\Lambda)$ smooth, birational to $\mathbb{P}\mathrm{Ext}^1(\Lambda, \mathcal{O}) = \mathbb{P}^{3g-3}$
- $\deg \Lambda$ = even

  $SU_C(2,\Lambda)$ is Gorenstein, has rational singularities

  rationality unknown (for $g \geq 3$)

Unirational: $\mathbb{P}^{3g-2} \dashrightarrow SU_C(2,\Lambda)$ generic fibers $\mathbb{P}^1$

Th (Narasimhan) det odd $\Rightarrow$
$\exists$ Poincaré vectorbundle $\mathcal{E}$ on $C \times SU_C(2,\Lambda)$ and
$\Phi_{\mathcal{E}} : D^b(C) \to D^b(SU_C(2,\Lambda))$ is fully faithful.

- In particular, C can be reconstructed as a moduli space of vector bundles on $SU_C(2,\Lambda)$: Torelli theorem!

Later, I will give a simple proof of Narasimhan's theorem, which can be extended to construct a full S.O.D. of $D^b(SU_C(2,\Lambda))$ conjectured by Narasimhan and Belmans-Galkin-Mukhopadhyay as well as of BPS categories for the even determinant (proved in collaboration w. S. Torres & B. Sink)

How about surfaces? Let $X = K3$.

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$H^1(X,\mathcal{O}_X)=0$ does not cause problems $\mathrm{Pic}^0 X = \{\mathcal{O}_X\}$

.... But $H^2(X,\mathcal{O}_X)\cong k$ does!

Indeed, take $E\in D^b(X)$ with $\mathrm{Ext}^j(E,E)=\begin{cases}0 & j<0\\ 1 & j=0\end{cases}$

Consider the composition

$$\mathrm{Ext}^1(E,E)\otimes\mathrm{Ext}^1(E,E)\xrightarrow{m}\mathrm{Ext}^2(E,E)\cong\mathrm{Hom}(E,E)^*=k$$

Fact (Mukai) m is skew-symmetric & non-degenerate

$\Rightarrow$ • moduli are unobstructed but also

• Hyperkähler!

How to find Fano subvarieties in HK varieties?

The restriction of $\Omega\in H^0(HK,\Omega^2)$ to a Fano subvariety vanishes $\Rightarrow$ Fano subvarieties are isotropic

Th (Aravena) If $\operatorname{Pic} K3 = \mathbb{Z}$, genus is even $\Rightarrow$

- $\exists$ Bridgeland stability condition σ with HK moduli space $M(\sigma)$ (birational to the Beauville-Mukai system)
- $\exists$ Fano Lagrangian $Y \subset M(\sigma)$
- $\exists$ Poincaré family $\mathcal{E}$ on $X \times Y$ of σ-stable objects

and the functor $\Phi_{\mathcal{E}} : D^b(X) \to D^b(Y)$ is fully faithful

What is the next obstruction to existence of Fano moduli?