Exercise in homological mirror symmetry

joint work with Yanki Lekili

Two players:

- **Affine toric surface X**

$$\operatorname{Spec} \mathbb{C}[x,y]^{\mu_r} = \frac{1}{r}(1,a) \qquad (r,a)=1$$

$$\zeta \in \mu_r, \quad \zeta \curvearrowright \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^a \end{bmatrix}$$

- **Non-commutative r-dimensional algebra R**
  - basis $\omega_i \qquad i \in \mathbb{Z}_r$
  - $\omega_i \cdot \omega_j = \begin{cases} \omega_{i+j} & (\text{modulo } r) \\ \text{or } 0 & ??? \end{cases}$

How to decide which is which?

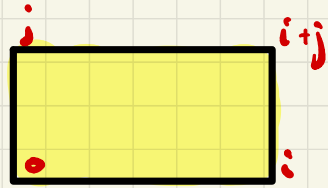
Let $b \equiv a^{-1} \bmod r$

Example: $r=7 \quad a=3 \quad b=5$

1	3	5	•	2	4	6	1	3
•	2	4	6	1	3	5	•	2
6	1	3	5	•	2	4	6	1
5	•	2	4	6	1	3	5	•
4	6	1	3	5	•	2	4	6
3	5	•	2	4	6	1	3	5
2	4	6	1	3	5	•	2	4
1	3	5	•	2	4	6	1	3
0	2	4	6	1	3	5	•	2

$+1 \uparrow$

$\xrightarrow[-b]{}$



$\omega_i \cdot \omega_j = \omega_{i+j} \Leftrightarrow$ no orange dots in the rectangle

$$\omega_i \cdot \omega_0 = \omega_0 \cdot \omega_i = \omega_i \qquad (\omega_0 = 1)$$

Other non-trivial products:

$\omega_2 \omega_1 = \omega_3$

$\omega_2^2 = \omega_4$

$\omega_2 \omega_3 = \omega_5$

$\omega_2 \omega_4 = \omega_6$

$\omega_4 \cdot \omega_1 = \omega_5$

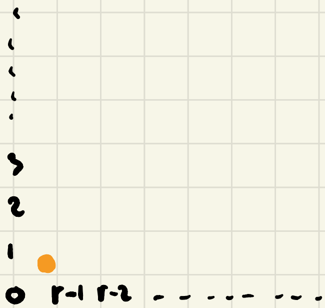
$\omega_4 \cdot \omega_2 = \omega_6$

Exercise: R is an associative algebra

Observation:

R is commutative in two cases:

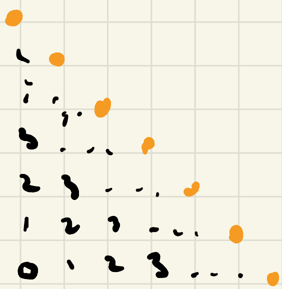
- $a=1 \Rightarrow b=1 = -b=-1$



$$w_iw_j = 0 \quad \forall \; i,j = 1,\dots,r-1$$

$$R = \mathbb{C}[w_1,\dots,w_{r-1}]/w_iw_j=0$$

- $a=b=-1 \Rightarrow -b=1$



$$w_i = w_1^{\,i} \qquad i=1,\dots,r-1$$

$$R = \mathbb{C}[w_1]/w_1^r=0$$

Theorem 1

(Lekili-T) R is isomorphic to the Kalck-Karmazyn algebra of X

[take as a definition for now]

(Kalck–Karmazyn) $D_{sing}(X) \simeq D_{sing}(R)$

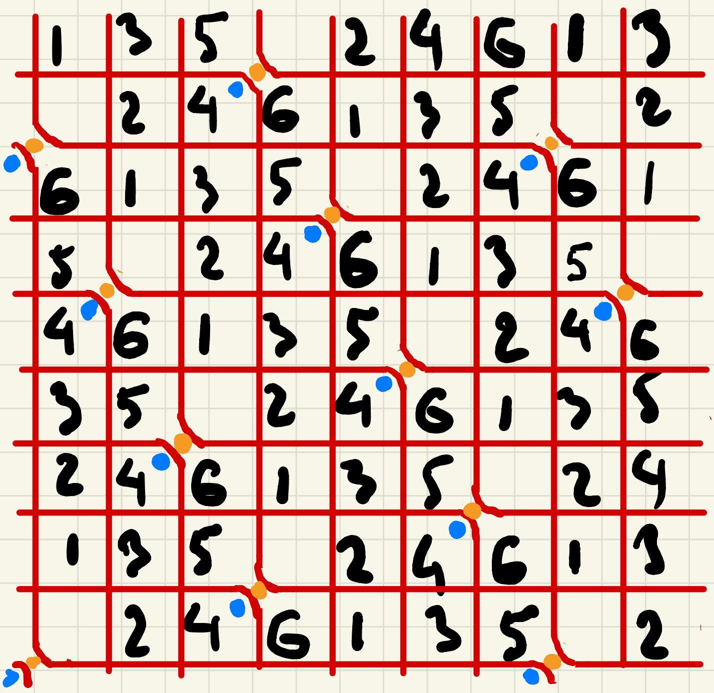
$$\begin{array}{ccc} D_{sing}(X) & \simeq & D_{sing}(R) \\ \| & & \| \\ D^b(X)/\mathrm{Perf}(X) & & D^b(\mathrm{mod}\text{-}R)/\mathrm{Perf}(R) \end{array}$$

[invert $\forall s$ $A \xrightarrow{s} B \to C \to$ for $C \in \mathrm{Perf}(X)$]

Let me sketch the proof and then generalize to deformations.

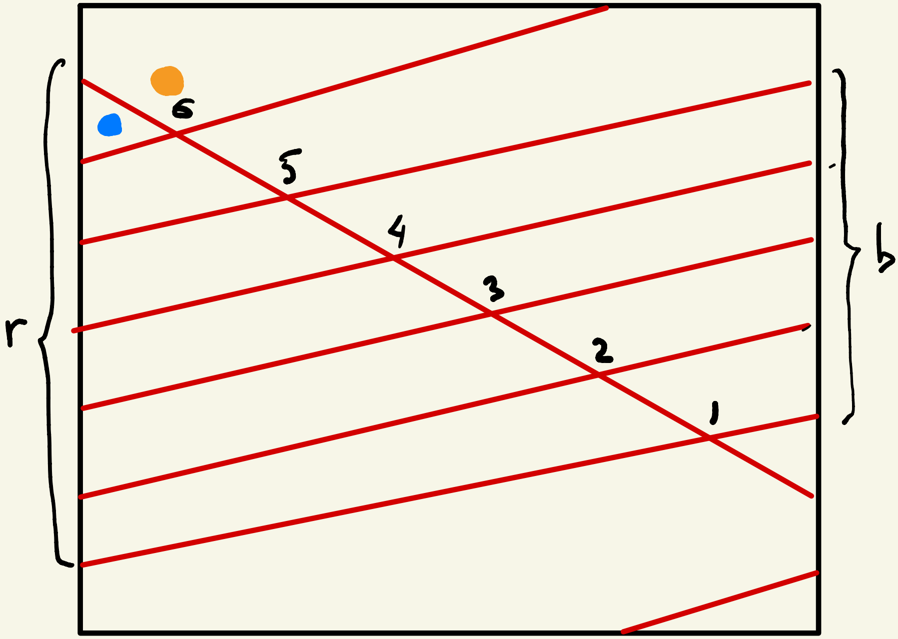
We start with some symplectic geometry

(1) Transform grid into curves in $\mathbb{R}^2$:



Interpret orange dots as "punctures" and add blue punctures to the left of the orange ones

(2) Exploit periodicity under $\mathbb{Z}^2 < \mathbb{R}^2$: project to 2-torus $\mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}$



Many curves in $\mathbb{R}^2$ become one connected closed curve $\mathbb{K} \subset \mathbb{T}$

(3) Perform a Floer homology calculation:

$\mathbb{T}_2$ symplectic torus with 2 punctures

$\exists$ exact symplectic form $\omega = d\lambda$

$\mathcal{F}(T_2)$ exact Fukaya category

as defined in Seidel's book:

objects = connected, compact, exact Lagrangians $\mathbb{L}$

($\exists \lambda|_{\mathbb{L}}$ exact)

with extra "brane data" - ignore

Every closed curve is homotopic to a unique exact Lagrangian upto a Hamiltonian isotopy

$\Rightarrow$ we can draw random curves and pretend they are Lagrangian

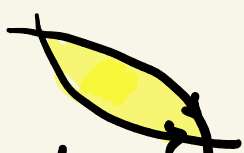
In particular, $\mathbb{K} \in \mathcal{F}(T_2)$

Lemma (Lekili-T.) $R = \mathrm{End}_{\mathcal{F}(T_2)} \mathbb{K}$

In general,

- $\mathrm{End}_{\mathcal{F}(T_2)} \mathbb{K} = \mathbb{C}\, \mathrm{Id} + \sum\limits_{\substack{\text{self-intersection} \\ \text{points of } \mathbb{K}}} \mathbb{C}\, \omega_i$

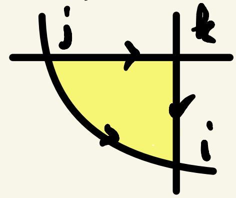
The sum is direct if there are no holomorphic bigons with



- true in our case!

Lagrangian boundary (so w/o punctures)

- Composition of endomorphisms is controlled by holomorphic triangles with Lagrangian boundary



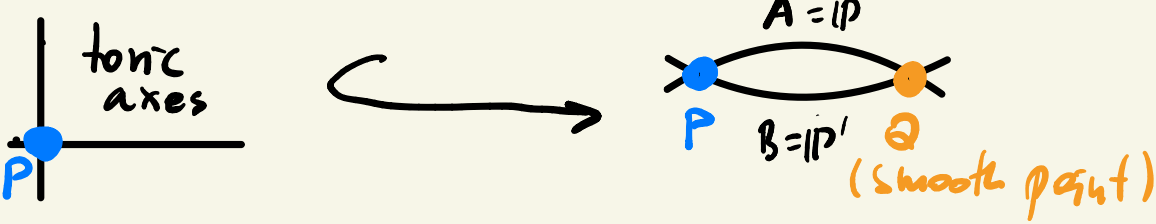
$$w_i w_j = \sum_{\text{triangles}} \pm w_k$$

But this just gives back our definition of R! $\square$

Next, let's do some algebraic geometry

(1) Compactify:

affine surface X $\quad$ proj. surface $\overline{X}$



$\frac{1}{r}(1,a)$

$E = A \cup B \in |-K|$

Exercise: $\overline{X}$ exists

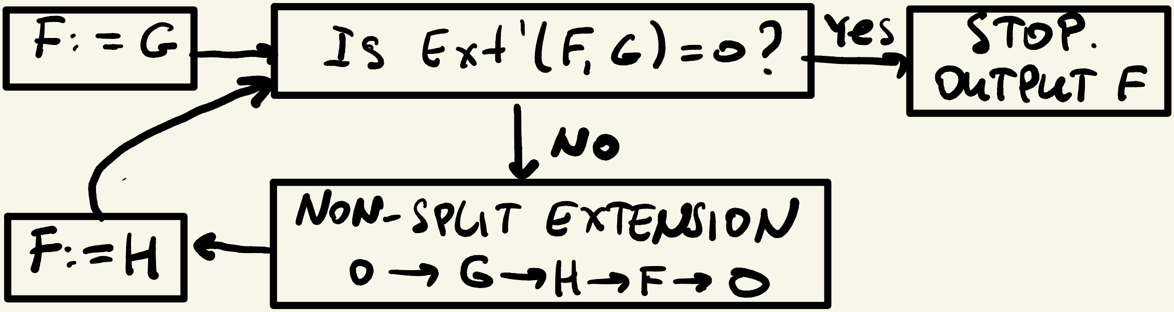
(2) Construct a vector bundle on $\bar{X}$:

Kawamata NC deformations:

X proj. variety, $G \in \mathrm{Coh}(X)$

(or G is an element of some abelian category)

Run an **algorithm**:



Typically, this algorithm does not terminate but if it does, the output does not depend on choices made.

Th (Kawamata, Karmazyn–Kuznetsov–Shinder)

The maximal iterated extension of the divisorial sheaf $\mathcal{O}(-A) \in \mathrm{Coh}\, X$ exists & is a vector bundle F of rank r called **Kawamata vector bundle**

Note: $\mathrm{Ext}^1(F,F)=0$ almost by definition

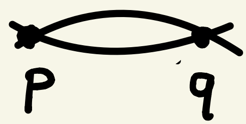
Easy Bonus: $\mathrm{Ext}^2(F,F)=0$

Almost an exceptional vector bundle

(3) Compute the endomorphism ring

Theorem (Lekili–T) $\mathrm{End}(F) \cong R$

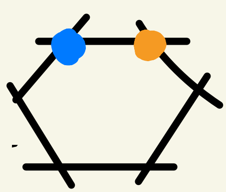
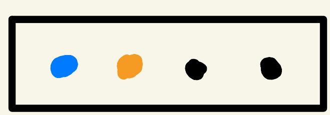
Proof • Reduce dimension:

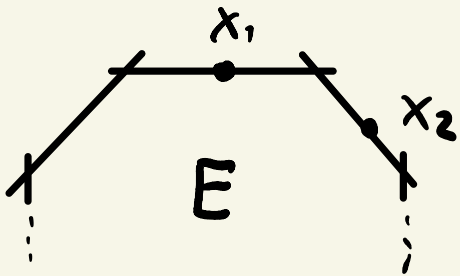
$E =$  $\subset W$

vector bundle on the curve! $F|_E$

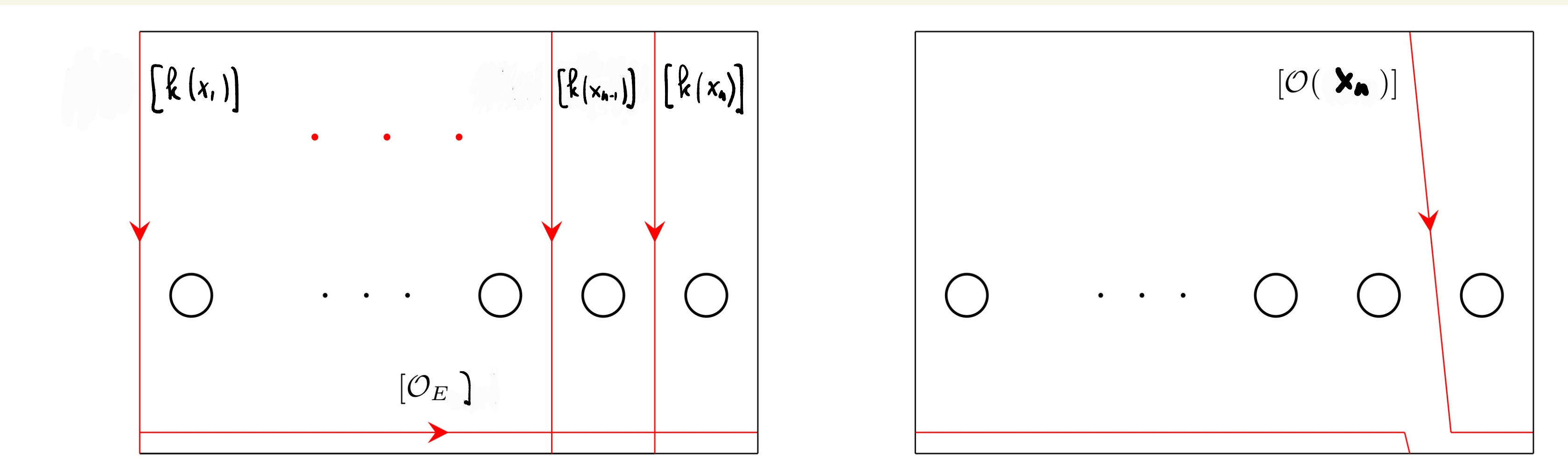
(Exercise) $\mathrm{End}(F) \cong \mathrm{End}(F|_E)$

• Use Homological mirror symmetry (Lekili–Polishchuk)

$\mathrm{Perf}\Big($  $\Big) \simeq \mathcal{F}\Big($  $\Big)$



Some mirrors :



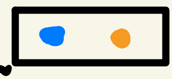
$0 \to \mathcal{O}_E \to \mathcal{O}_E(x_n) \to k(x_n) \to 0 \rightsquigarrow$ Dehn twists

Perf E $\qquad\qquad$ $\mathcal{F}(\pi)$

- **Identify the mirror**

Kawamata v.b. $\qquad$ Kawamata Lagrangian

$$\mathcal{F}l_E \longleftrightarrow \mathbb{K}$$

$\mathcal{F}l_E \subset$ Perf() $\qquad$ $\mathbb{K} \subset \mathcal{F}$()

In particular, $\mathrm{End}(\mathcal{F}l_E) \cong \mathrm{End}(\mathbb{K}) \cong R$ $\blacksquare$

Now we can invoke various results:

Categorical absorption

Theorem (Kuznetsov–Karmazyn–Shinder)

$D^b(\overline{X}) = \langle \mathcal{A}, \mathcal{B} \rangle$ SOD

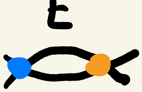
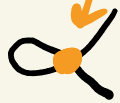
$\mathcal{A} \cong D^b(R\text{-mod})$ $\qquad$ $\mathcal{B} \subset \mathrm{Perf}(X)$

Corollary (proved earlier by Kalck–Karmazyn)

$$D_{sing}(X) = D_{sing}(\overline{X}) \cong D_{sing}(R)$$

Next: let's deform!

keep the node to use exact Fukaya category

$(\mathcal{E}\subset\mathfrak{X})$ $\qquad$ E  $\subset \overline{X}$ $\qquad$  $\subset Y$ (smooth surface)

$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$

$\mathbb{A}^1_t \qquad\qquad t=0 \qquad\qquad t\neq 0$

Theorem (Kollár–Shepherd-Barron) Def_X has smooth irreducible components indexed by certain partial resolutions of X

Exercise: The same is true for $\mathrm{Def}_{\overline{X}}$ and $\mathrm{Def}_{E\subset\overline{X}}$

Theorem (T.-Urzúa)

- Kawamata bundle F deforms to $\mathcal{F}$ on $\mathcal{X}$
- $\mathcal{R} = \mathrm{End}(\mathcal{F})$ is a flat deformation of R
- Categorical absorption

$$D^b(\mathcal{X}) = \langle D^b(\mathcal{R}), \mathcal{B}\rangle, \quad \mathcal{B}\subset \mathrm{Perf}(\omega)$$

- If we choose a general smoothing within some irreducible component of $\mathrm{Def}_{E\subset\overline{X}}$ then R_t is a hereditary algebra

(which depends on that component)

$\Leftrightarrow \mathrm{Ext}^2(X, Y) = 0 \quad X, Y \in R_t\text{-mod}$

$\Leftrightarrow R_t$ is Morita-equivalent to the path algebra of an acyclic quiver without relations

Example $\frac{1}{4}(1,3)$ $\quad r=4 \quad a=3$

$b=3 \quad -b=1$

$$\frac{1}{4}(1,3) = (uv = w^4)$$

versal deformation: $uv = w^4 + a_1 w^2 + a_2 w + a_3$

Kalck-Karmazyn algebra

$$R = k[t]/t^4$$

versal deformation: $k[t]/t^4 + a_1 t^2 + a_2 t + a_3$

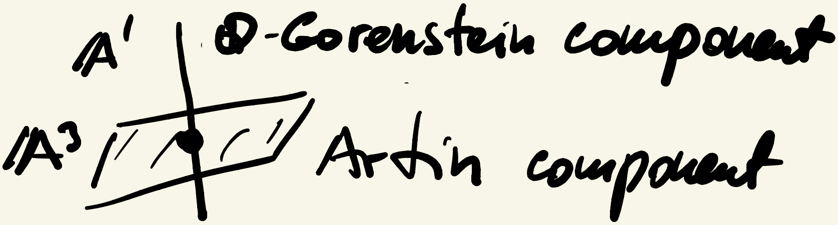
no multiple roots $\Rightarrow$ $k \oplus k \oplus k \oplus k$

quiver $\bullet \quad \bullet \quad \bullet \quad \bullet$

Example $\frac{1}{4}(1,1)$

By Pinkham (73) this singularity has 2 types of deformations,

Def X



Kalck–Karmazyn algebra

$$R = k[w_1, w_2, w_3] / w_i w_j = 0$$

Gabriel (73) classified its deformations

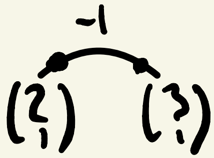
Artin deformation of X	$\mathbb{Q}$-Gorenstein deformation of X
$\rightsquigarrow$	$\rightsquigarrow$
deformation of R to Path( )	deformation of R to $\mathrm{Mat}_2(\mathbb{C})$

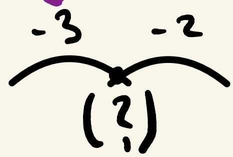
(There are many other deformations of R)

Example from K-SB $\frac{1}{19}(1,7)$ has 3 M-resolutions

(self-intersections on the resolution)

minimal resolution (self-intersection)

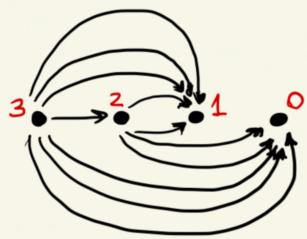


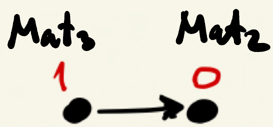


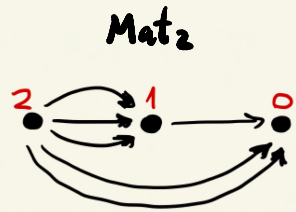
$$\frac{19}{7} = 3 - \frac{1}{4 - \frac{1}{2}}$$

$$\binom{n}{a} = \frac{1}{n^2}(1, na-1)$$ Wahl singularity

Corresponding quivers:







$9+4+6=19$

$1+4+1+2+6+2+3=1$

$\dim R = \dim \hat{R} = 19$

Th (Orlov) $D^b(\text{any quiver}) \xrightarrow{\text{fully faithful}} D^b(\text{variety of large dimension})$

Rk Most quivers are not embeddable in D^b(surface) (strong restrictions on Mukai lattice studied by Belmans-Raedschelders)

Corollary A lot of quivers do embed by our construction, in fact enough to prove conjectures of Belmans-Raedschelders on possible embeddings

Now that we understand $R = R_0$ and its deformation R_t, $t \neq 0$, How about the family of algebras $\mathcal{R}$?

Th (Lekili-T.) Explicit formulas for $k[t]$-algebras $\mathcal{R}$.

Example

Wahl singularity $\frac{1}{n^2}(1, nq-1)$ $(n,q)=1$

Precisely $\frac{1}{r}(1,a)$ admitting
one-parameter $\mathbb{Q}G$ smoothing

$$\overline{X} \longrightarrow Y$$

$t=0 \qquad t\neq 0$

rank n^2

Th (Kawamata) $\mathcal{F}_t$ for $t\neq 0$
splits as $\mathcal{F}_t \cong H^{\oplus n}$, where
H is a rank n exceptional vector bundle
introduced by Hacking.

$$\Rightarrow \mathcal{R}_t = \mathrm{End}(\mathcal{F}_t) \cong \mathrm{Mat}_n(k)$$

By Tsen's Theorem, $\mathcal{R}$ is isomorphic
to a matrix order:

- $\mathcal{R} \subset \mathrm{Mat}_{n\times n}(k(t))$
- $\mathcal{R}$ is a $k[t]$-subalgebra
- $\mathcal{R}$ is a free $k[t]$-submodule of rank n^2

Th (Lekili–T) Explicit formulas for $\mathcal{R}$:

Examples

$\frac{1}{4}(1,1)$
$$\begin{bmatrix} a_0 & ta_3 \\ -ta_1 & -ta_2 + a_0 \end{bmatrix} \qquad a_i \in k[t]$$

$\frac{1}{9}(1,2)$
$$\begin{bmatrix} -t^2a_6 + a_0 & ta_4 + a_1 & t^2a_8 + ta_5 \\ -t^3a_8 & a_0 & t^2a_7 \\ -ta_1 & -ta_2 & -t^2a_6 - ta_3 + a_0 \end{bmatrix}$$

Approach for explicit calculation:

(1) Factorization of the deformation problem:

$$\begin{tikzcd}
\mathrm{Def}_{E\subset X} \arrow[r, hook] \arrow[dd] & \mathrm{Def}\, R \\
(\mathcal{E} \subset \mathfrak{X}) \arrow[r, mapsto] & \mathrm{End}\, \mathcal{F}|_{\mathcal{E}} \\
\mathrm{Def}_{F|_E, \mathcal{E}} &
\end{tikzcd}$$

(deformations of $F|_E$ to $\mathcal{E}$)

However, for a typical deformation $\mathcal{V}$ of $F|_E$, $\dim \operatorname{End}(\mathcal{V}) < \dim \operatorname{End} F|_E$
$$\dim \operatorname{End} F|_E = \dim R$$

Define $\mathrm{Def}^{\circ}_{F|_{E},\,\mathcal{E}} \subset \mathrm{Def}_{F|_{E},\,\mathcal{E}}$ by requirement $\dim \operatorname{End} \mathcal{V} = r$ (maximal possible).

Factorization

$$\begin{array}{ccccc} \mathcal{F} & \longmapsto & V := \mathcal{F}|_{\mathcal{E}} & \longmapsto & \operatorname{End}(V) \\ \mathrm{Def}_{E\subset X} & \longrightarrow & \mathrm{Def}^{\circ}_{F|_{E},\,\mathcal{E}} & \longrightarrow & \mathrm{Def}_R \end{array}$$

From mirror symmetry:

Th (Lekili–T.) Explicit description of $\mathrm{Def}^{\circ}_{F|_{E},\,\mathcal{E}}$ and the family of algebras $\operatorname{End}(V)$ over it

Apriori one expects $\mathrm{Def}^{\circ}_{F|_{E},\,\mathcal{E}}$ to be big but

Conjecture (verified for $r \leq 32$)

$$\mathrm{Def}_{E\subset \bar{X}} \simeq \mathrm{Def}^{\circ}_{F|_{E},\,\mathcal{E}}$$

Mirror Symmetry for $\mathcal{E}$ (Lekili-Polishchuk)

$$\operatorname{Perf} \mathcal{E} \cong \operatorname{Fuk}(\Pi_1, \{s\})$$

relative Fukaya

orange puncture

blue divisor

In description of compositions of morphisms, every polygon u contributes $s^{\operatorname{mult}(u, \{s\})}$

$\operatorname{Def}^{\circ}_{F|_{\mathcal{E}, \mathcal{E}}}$ is computed on $\operatorname{Fuk}(\Pi_1, \{s\})$ using the formalism of bounding cochains of Fukaya-Oh-Ohta-Ono.