

HOMOLOGICAL METHODS

JENIA TEVELEV

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§0. SYLLABUS

Calculations of various homology and cohomology groups are ubiquitous in mathematics: they provide a systematic way of reducing difficult geometric (=non-linear) problems to linear algebra. I will attempt to develop homological machinery starting with the basic formalism of complexes, exact sequences, spectral sequences, derived functors, etc. in the direction of understanding cohomology of coherent sheaves on complex algebraic varieties. There will be detours to algebraic topology and commutative algebra. Prerequisites are graduate algebra, manifolds (for a detour into algebraic topology), and complex analysis. Some familiarity with algebraic varieties (a one-semester course or a reading course should be enough). Algebraic varieties will not appear in the first half of the course, and, in any case, we will study sheaves from scratch.

I won't follow any particular book, but I will borrow heavily from

S. Weibel, *An Introduction to Homological Algebra*

S. Gelfand, Yu. Manin, *Methods of Homological Algebra*

R. Bott, L. Tu, *Differential Forms in Algebraic Topology*

D. Eisenbud, *Commutative Algebra with a view towards Algebraic Geometry*

V.I. Danilov, *Cohomology of Algebraic Varieties*

S. Lang, *Algebra*

D. Huybrechts, *Fourier–Mukai transforms in Algebraic Geometry*

C. Voisin, *Hodge Theory and Complex Algebraic Geometry*

Grading will be based on 6 biweekly homework sets. Each homework will have a two-week deadline and individual problems in the homework will be worth some points (with a total of about 30 points for each homework). The passing standard (for an A) will be 90 points by the end of the

semester. The idea is to make homework assignments very flexible. Beginning students are advised to solve many “cheap” problems to gain experience. If you feel that you already have some familiarity with the subject, pick a few hard “expensive” problems.

Homework problems can be presented in two ways. An ideal method is to come to my office and explain your solution. If your solution is correct, you won’t have to write it down. If your solution does not work, I will give you a hint. Problems not discussed orally will have to be written down and turned in at the end of the two-week period. All office hours will be by appointment. I will ask you to sign up for an individual weekly 30 minutes slot. You can also team up and convert two adjacent short slots into one hour-long joint meeting. I am available on M, W from 2:00-6:00 and Tu, Th 3:30-6:00. Please e-mail me your preferred meeting time as soon as possible. During the semester, please e-mail me in advance if by whatever reason you won’t make it to our weekly meeting.

§1. COMPLEXES. LONG EXACT SEQUENCE. JAN 23.

Fix a ring R , not necessarily commutative.

The category $R\text{-mod}$ of left R -modules: morphisms are R -linear maps.

Examples: $\mathbb{Z}\text{-mod} = \mathbf{Ab}$ (Abelian groups), $k\text{-mod} = \mathbf{Vect}_k$ (k -vector spaces). For algebraic geometry: $R = k[x_1, \dots, n]$ (or its quotient algebra, localization, completion, etc.). For complex geometry: $R = \mathcal{O}_{hol}(U)$ for an open subset $U \subset \mathbb{C}^n$. Representation theory: rings like $R = k[G]$ ($R\text{-mod} = k$ -representations of a group G), etc.

$R\text{-mod}$ is an example of an Abelian category, which we are going to discuss later, after we have more examples. Constructions of homological algebra apply to any Abelian category, for example to the category of sheaves on a topological space, \mathcal{O}_X -modules on a ringed space, etc (to be discussed later as well). Things you can compute in an Abelian category but not in a general category:

- 0 object;
- $f + g$ for two morphisms $f, g : X \rightarrow Y$;
- $X \oplus Y$;
- $\text{Ker}(f), \text{Coker}(f), \text{Im}(f)$ for a morphism $f : A \rightarrow B$;
- the first isomorphism theorem $A/\text{Ker}(f) \simeq \text{Im}(f)$.

1.1. DEFINITION. • (Cochain) complexes

$$\dots \xrightarrow{d_{i-1}} C^i \xrightarrow{d_i} C^{i+1} \xrightarrow{d_{i+1}} \dots, \quad d_{i+1} \circ d_i = 0 \quad \text{for any } i.$$

- Cohomology groups: $H^i = \text{Ker } d_{i+1} / \text{Im } d_i$.
- Exact sequences: $\text{Ker } d_{i+1} = \text{Im } d_i$.

One can also consider chain complexes and homology groups.

1.2. LEMMA (Snake Lemma). *Given a commutative diagram in $R\text{-mod}$*

$$\begin{array}{ccccccc} X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & 0 \\ & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 \\ 0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 \end{array}$$

with exact rows, one has an exact sequence

$$\text{Ker}(f_1) \rightarrow \text{Ker}(f_2) \rightarrow \text{Ker}(f_3) \xrightarrow{\delta} \text{Coker}(f_1) \rightarrow \text{Coker}(f_2) \rightarrow \text{Coker}(f_3)$$

where all maps are induced by maps in exact sequences except for δ (called connecting homomorphism), which is defined as follows: lift $\alpha \in \text{Ker}(f_3)$ to $\beta \in X_2$, take its image in Y_2 , lift it to Y_1 , take a coset in $\text{Coker}(f_1)$.

Proof. Diagram chasing. Don't forget to show that δ is R -linear. \square

Maps of complexes

$$f^\bullet : X^\bullet \rightarrow Y^\bullet$$

are maps $f^i : X^i \rightarrow Y^i$ that commute with differentials. They induce maps of cohomology

$$H^i(X) \rightarrow H^i(Y) \quad \text{for any } i.$$

1.3. LEMMA (Long Exact Sequence). *Short exact sequence of complexes*

$$0 \rightarrow X^\bullet \rightarrow Y^\bullet \rightarrow Z^\bullet \rightarrow 0$$

induces a long exact sequence of cohomology

$$\dots \xrightarrow{\delta} H^i(X) \rightarrow H^i(Y) \rightarrow H^i(Z) \xrightarrow{\delta} H^{i+1}(X) \rightarrow H^{i+1}(Y) \rightarrow \dots$$

Proof. Follows from (and is equivalent to) the Snake Lemma. \square

§2. CATEGORIES AND FUNCTORS. JAN 25.

A reminder about categories and functors: a *category* C has a set of objects $\text{Ob}(C)$ and a set of morphisms $\text{Mor} = \coprod_{X,Y \in \text{Ob}(C)} \text{Mor}(X,Y)$, i.e. each morphism has a source and a target. It also has a composition law

$$\text{Mor}(X,Y) \times \text{Mor}(Y,Z) \rightarrow \text{Mor}(X,Z), \quad (f,g) \mapsto g \circ f$$

and an identity morphism $\text{Id}_X \in \text{Mor}(X,X)$ for each object X . Two axioms have to be satisfied:

- $f \circ \text{Id}_Y = f = f \circ \text{Id}_X$ for each $f \in \text{Mor}(X,Y)$.
- Composition is associative: $(f \circ g) \circ h = f \circ (g \circ h)$.

2.1. EXAMPLE. $R\text{-mod}$, **Sets**, **Top** (topological spaces as objects and continuous maps between them as morphisms).

A (covariant) *functor* $F : C_1 \rightarrow C_2$ from one category to another is a function $F : \text{Ob}(C_1) \rightarrow \text{Ob}(C_2)$ and functions $F : \text{Mor}(X,Y) \rightarrow \text{Mor}(F(X),F(Y))$. Two axioms have to be satisfied:

- $F(\text{Id}_X) = \text{Id}_{F(X)}$;
- $F(f \circ g) = F(f) \circ F(g)$.

A *contravariant functor* $F : C_1 \rightarrow C_2$ is a function $F : \text{Ob}(C_1) \rightarrow \text{Ob}(C_2)$ and functions $F : \text{Mor}(X,Y) \rightarrow \text{Mor}(F(Y),F(X))$. The first axiom is the same, the second changes to $F(f \circ g) = F(g) \circ F(f)$.

2.2. EXAMPLE. . For each integer i there is an i -th cohomology functor

$$H^i : \mathbf{Kom}(R\text{-mod}) \rightarrow R\text{-mod}.$$

Notice that this means that not only we can compute $H^i(C^\bullet)$ of any complex but also that for each map of complexes $C^\bullet \rightarrow D^\bullet$, there is an induced

map $H^i(C^\bullet) \rightarrow H^i(D^\bullet)$, which satisfies two axioms above. As a general rule, morphisms are more important than objects (and functors are more important than categories).

§3. SIMPLICIAL HOMOLOGY.

A *simplicial complex* X (not to be confused with its chain complex) is a picture glued from points, segments, triangles, tetrahedra, etc. Formally, X is a set of finite subsets of the vertex set V such that:

- $\{v\} \in X$ for any $v \in V$;
- if $S \in X$ then $T \in X$ for any $T \subset S$. We call T a *face* of S .

We are also going to assume that V is well-ordered (for example $V = \{0, 1, 2, \dots, n\}$). A topological space $|X|$ (called a *geometric realization* of X) is defined as follows. Let $X_n \subset X$ be the set of subsets with $n + 1$ points. Let

$$\Delta_n = \{(x_0, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_i \geq 0, \sum x_i = 1\}$$

be the standard n -dimensional simplex (with $n + 1$ vertices). For each $i = 0, \dots, n$, let

$$\partial_n^i : \Delta_{n-1} \rightarrow \Delta_n \tag{3.1}$$

be the standard unique affine linear map which sends vertices of Δ_{n-1} to vertices of Δ_n (except for the i -th vertex). The mapping of vertices is given by a unique increasing function $\{0, 1, \dots, n - 1\} \rightarrow \{0, 1, \dots, n\} \setminus \{i\}$. As a set, $|X|$ is the disjoint union of all simplices $\coprod (X_n \times \Delta_n)$ modulo an equivalence relation generated by the relation

$$(T, p) \sim (S, \delta_n^i(p))$$

each time T is a subset of S obtained by dropping its i -th index (recall that all vertices are well-ordered). Topology on $|X|$ is defined as follows: a subset U is open if and only if its preimage in each copy of Δ_n is open. In other words, it is the weakest topology such that the map

$$\coprod_{n \geq 0} (X_n \times \Delta_n) \rightarrow |X|$$

is continuous.

Now let's fix an Abelian group A and define the chain group $C_n(X, A)$ to be the group of formal finite linear combinations

$$\sum_{x \in X_n} a(x)x, \quad a(x) \in A.$$

The chain differential is defined as follows:

$$\partial_n \left(\sum_{x \in X_n} a(x)x \right) = \sum_{x \in X_n} a(x)\partial_n(x),$$

and

$$\partial_n(\{i_0, \dots, i_n\}) = \sum_{j=0}^n (-1)^j \{i_0, \dots, \hat{i}_j, \dots, i_n\}$$

(Recall that a simplex is just a subset $\{i_0, \dots, i_n\} \subset V$, V is well-ordered, so I can always assume that $i_0 < \dots < i_n$).

3.2. LEMMA. ∂ is a differential, i.e. $\partial_{n-1} \circ \partial_n = 0$.

Proof. By linearity, it suffices to check that $\partial_{n-1} \circ \partial_n(\{i_0, \dots, i_n\}) = 0$. Observe that this is a linear combination of subsets of $\{i_0, \dots, i_n\}$ with $n - 1$ elements, where each subset appears twice, with opposite signs. \square

We define *simplicial homology groups* $H_i(X, A)$ as homology groups of the simplicial chain complex $C_\bullet(X, A)$ defined above.

Notice that these groups can be effectively computed. For example, consider a triangulation of the sphere S^2 given by the faces of the tetrahedron. This gives a chain complex $C_\bullet(X, \mathbb{Z})$. It looks as follows:

$$\mathbb{Z}^4 \rightarrow \mathbb{Z}^6 \rightarrow \mathbb{Z}^4$$

with maps given by very explicit matrices. The homology groups look as follows:

$$\mathbb{Z}, \quad 0, \quad \mathbb{Z}.$$

The price to pay: it is not clear that these groups are invariants of S^2 , i.e. that they do not depend on a choice of triangulation. In fact this is true and gives, among other things,

3.3. THEOREM (Euler). *For any triangulation of the sphere,*

$$\#vertices - \#edges + \#faces = 2. \quad (3.4)$$

This follows from a more general theorem, for which we need a definition.

3.5. DEFINITION. Let Γ be an Abelian group and let R be an R -module. An *Euler–Poincaré mapping* ϕ with values in Γ is a partially defined function that assigns to an R -module an element of Γ such that the following is true. For any exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0,$$

$\phi(B)$ is defined if and only if both $\phi(A)$ and $\phi(C)$ are defined. In this case we should have

$$\phi(B) = \phi(A) + \phi(C).$$

The most basic example is dimension, which is an Euler–Poincaré mapping for vector spaces. The definition above makes sense in any category with a concept of an exact sequence, i.e. in any Abelian category.

3.6. THEOREM (Euler–Poincaré). *Consider a bounded chain complex C_\bullet of R -modules (i.e. $C_k = 0$ for $k \ll 0$ or $k \gg 0$). Suppose $\phi(C_k)$ is defined for any k . Then $\phi(H_k)$ is defined for any k and we have*

$$\sum_k (-1)^k \phi(C_k) = \sum_k (-1)^k \phi(H_k).$$

This common quantity is called Euler characteristic. Analogous result holds for cochain complexes.

Proof. Follows from additivity of ϕ on exact sequence

$$0 \rightarrow \text{Ker } d_k \rightarrow C_k \rightarrow \text{Im } d_k$$

and

$$0 \rightarrow \text{Im } d_{k+1} \rightarrow \text{Ker } d_k \rightarrow H_k.$$

□

To prove Euler’s theorem on triangulations, simply notice that (3.4) is an Euler characteristic of a simplicial chain complex of the triangulation, which by Euler–Poincare theorem is equal to $1 - 0 + 1 = 2$.

§4. SINGULAR HOMOLOGY

Now we would like to define the homology theory called *singular homology* with coefficients in an arbitrary Abelian group A , which depends only on the topological space X , more precisely on its homotopy type (we will clarify this statement later). The construction is very similar to simplicial homology, except that we will allow chains of arbitrary simplices rather than only simplices appearing in the triangulation.

Let X_n be the set of all continuous maps $g : \Delta_n \rightarrow X$. Let $C_n(X, A)$ be the group of formal finite linear combinations

$$\sum_{g \in X_n} a(g)g$$

with coefficients in A . The differential is defined as follows:

$$\partial_n \left(\sum_{g \in X_n} a(g)g \right) = \sum_{g \in X_n} a(g)\partial_n(g),$$

and

$$\partial_n(g) = \sum_{i=0}^n (-1)^i g \circ \partial_n^i,$$

where ∂_n^i is given by (3.1). The same argument as in simplicial homology gives $\partial_{n-1} \circ \partial_n = 0$, and so we have a *singular chain complex* $C_\bullet(X, A)$ and *singular homology groups* $H_\bullet(X, A)$.

§5. FUNCTORIALITY OF SINGULAR HOMOLOGY. JAN 27

The singular chain complex $C_\bullet(X, A)$ and its homology groups $H_\bullet(X, A)$ depend on both X and A in a functorial way.

For example, for any continuous map $f : X \rightarrow Y$, we have an induced homomorphism

$$f_* : C_n(X, A) \rightarrow C_n(Y, A), \quad \sum_{x \in X_n} a(x)x \mapsto \sum_{x \in X_n} a(x)f \circ x$$

for each n . Recall that $x \in X_n$ is a continuous map $\Delta_n \xrightarrow{x} X$, and $f \circ x$ is just its composition with f . It is clear that f_* commutes with differentials and therefore induces a map of complexes $f_* : C_\bullet(X, A) \rightarrow C_\bullet(Y, A)$, and induced homomorphisms of Abelian groups $H_\bullet(X, A) \rightarrow H_\bullet(Y, A)$. In short, we have a functor

$$\mathbf{Top} \rightarrow \mathbf{Kom}(\mathbf{Ab}), \quad X \mapsto C_\bullet(X, A).$$

For example, suppose we have an embedding $i : Y \hookrightarrow X$ of topological spaces. Then we have a map, in fact clearly a monomorphism

$$i_* : C_\bullet(Y, A) \hookrightarrow C_\bullet(X, A).$$

We define a complex of *relative singular chains* $C_\bullet(X, Y; A)$ as a cokernel of i_* , i.e. $C_k(X, Y; A) = C_k(X, A)/C_k(Y, A)$ for each k .

Similarly, for each homomorphism $f : A \rightarrow B$, we have an induced homomorphism

$$C_n(X, A) \rightarrow C_n(X, B), \quad \sum_{x \in X_n} a(x)x \mapsto \sum_{x \in X_n} f(a(x))x,$$

for each n . These homomorphisms obviously commute with differentials, which means that they give a map of complexes $C_\bullet(X, A) \rightarrow C_\bullet(X, B)$ and a homomorphism of homology groups $H_\bullet(X, A) \rightarrow H_\bullet(X, B)$. In short, we have a functor

$$\mathbf{Ab} \rightarrow \mathbf{Kom}(\mathbf{Ab}), \quad A \mapsto C_\bullet(X, A).$$

For example, any short exact sequences of Abelian groups

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

gives a short exact sequence of singular chain complexes

$$0 \rightarrow C_\bullet(X, A) \rightarrow C_\bullet(X, B) \rightarrow C_\bullet(X, C) \rightarrow 0,$$

and an induced long exact sequence of homology

$$\dots \rightarrow H_k(X, A) \rightarrow H_k(X, B) \rightarrow H_k(X, C) \rightarrow H_{k-1}(X, A) \rightarrow \dots$$

§6. DE RHAM COHOMOLOGY.

As an example of cohomology theory, let's review basics of de Rham cohomology. With each smooth manifold M , we are going to associate its de Rham complex of \mathbb{R} -vector spaces

$$\Omega^0(M, \mathbb{R}) \xrightarrow{d} \Omega^1(M, \mathbb{R}) \xrightarrow{d} \Omega^2(M, \mathbb{R}) \xrightarrow{d} \dots$$

For each smooth map $f : M \rightarrow N$, and each k , we have a pull-back linear map $f^* : \Omega^k(N, \mathbb{R}) \rightarrow \Omega^k(M, \mathbb{R})$. Altogether, this gives a contravariant functor

$$\mathbf{Mflds} \rightarrow \mathbf{Kom}(\mathbf{Vect}_R)$$

from the category of smooth manifolds to the category of complexes of vector spaces. Taking cohomology of de Rham complexes, we get de Rham cohomology groups $H_{dR}^k(M, \mathbb{R})$ for each $k \geq 0$.

Now the details. Since smooth manifolds of dimension n are glued from open subsets of \mathbb{R}^n , it is useful to consider this situation first, where we can and will work in coordinates.

For an open subset $U \subset \mathbb{R}^n$, we define $\Omega^k(U, \mathbb{R}) = C^\infty(U) \otimes_{\mathbb{R}} \Lambda^k(\mathbb{R}^n)^\vee$ as a vector space of k -linear differentiable forms

$$\omega = \sum_{1 \leq i_1 < \dots < i_k \leq n} f_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

Here $f_{i_1 \dots i_k} \in C^\infty$ is a smooth function for each subset i_1, \dots, i_k . Symbols dx_i satisfy standard relations of the exterior algebra:

$$dx_i \wedge dx_j = -dx_j \wedge dx_i,$$

which makes $\Omega^\bullet(U, \mathbb{R})$ into a *graded \mathbb{R} -algebra*¹ with respect to the exterior product \wedge . And of course we have an “exterior” differential

$$d : \Omega^k(U, \mathbb{R}) \rightarrow \Omega^{k+1}(U, \mathbb{R})$$

$$\sum_{1 \leq i_1 < \dots < i_k \leq n} f_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k} \mapsto \sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\sum_{j=1}^n \frac{\partial f_{i_1 \dots i_k}}{\partial x_j} dx_j \right) dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

Clairaut’s Theorem (equality of mixed partials) implies that $d^2 = 0$, so $\Omega^\bullet(U, \mathbb{R})$ is also a complex, called *de Rham complex*. The differential and exterior multiplication are related by the following equation:

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2 \quad \text{for } \omega_1 \in \Omega^k(U, \mathbb{R}), \omega_2 \in \Omega^l(U, \mathbb{R}).$$

This structure can be formalized as follows:

6.1. DEFINITION. A differentiable graded algebra (DGA) is a graded k -algebra

$$A = \bigoplus_{n=0}^{\infty} A^n \text{ equipped with a differential } d \text{ such that}$$

- (A^\bullet, d) is a cochain complex;
- $d(w_1 \cdot w_2) = dw_1 \cdot w_2 + (-1)^k w_1 \cdot dw_2$ if $w_1 \in A^k, w_2 \in A^l$.

So $\Omega^\bullet(U, \mathbb{R})$ is a DGA.

Now let’s discuss functoriality: given open sets $U \subset \mathbb{R}^n$ (with coordinates x_1, \dots, x_n), $V \subset \mathbb{R}^m$ (with coordinates y_1, \dots, y_m), and a smooth map $\alpha : U \rightarrow V$, we define the pullback map

$$\alpha^* : \Omega^k(V, \mathbb{R}) \rightarrow \Omega^k(U, \mathbb{R}),$$

$$\sum_{1 \leq i_1 < \dots < i_k \leq m} f_{i_1 \dots i_k} dy_{i_1} \wedge \dots \wedge dy_{i_k} \mapsto \sum_{1 \leq i_1 < \dots < i_k \leq n} (f_{i_1 \dots i_k} \circ g) d(y_{i_1} \circ g) \wedge \dots \wedge d(y_{i_k} \circ g),$$

where for any function h on U (for example $y_i \circ g$), dh denotes its differential from calculus

$$dh = \sum_i \frac{\partial h}{\partial x_i} dx_i$$

It is not hard to show that this gives a contravariant functor for each k

$$\left\{ \begin{array}{l} \text{open subsets of a vector space,} \\ \text{smooth maps} \end{array} \right\} \rightarrow \mathbf{DGA}_{\mathbb{R}},$$

$$U \mapsto \Omega^\bullet(U, \mathbb{R}), \quad [U \xrightarrow{\alpha} V] \mapsto [\Omega^\bullet(V, \mathbb{R}) \xrightarrow{\alpha^*} \Omega^\bullet(U, \mathbb{R})].$$

Indeed, it immediately follows from definitions that the exterior multiplication and the exterior differential are preserved by pull-back. The most nontrivial part is that

$$(f \circ g)^* = g^* \circ f^*$$

¹A ring S decomposed into a direct sum $S = \bigoplus_{i=0}^{\infty} S_i$ of Abelian subgroups is called graded if $S_i \cdot S_j \subset S_{i+j}$. A ring S with a ring homomorphism $k \rightarrow S$ is called a k -algebra.

for a composition $U \xrightarrow{g} V \xrightarrow{f} W$ of smooth maps. This is a Chain Rule from calculus.

Now we extend our de Rham functor to manifolds. A manifold M has an atlas $M = \cup_{\alpha} U_{\alpha}$, where each U_{α} is isomorphic to an open subset of \mathbb{R}^n . Fix this isomorphism. Each intersection $U_{\alpha} \cap U_{\beta}$ is isomorphic to an open subset of \mathbb{R}^n in two different ways, via inclusions

$$i_{\alpha\beta}^{\alpha} : U_{\alpha} \cap U_{\beta} \hookrightarrow U_{\alpha}, \quad i_{\alpha\beta}^{\beta} : U_{\alpha} \cap U_{\beta} \hookrightarrow U_{\beta}$$

Fix one of these isomorphisms. Then we define

$$\Omega^k(M, \mathbb{R}) = \left\{ (\omega_{\alpha}) \in \prod_{\alpha} \Omega^k(U_{\alpha}, \mathbb{R}) \mid (i_{\alpha\beta}^{\alpha})^* \omega_{\alpha} = (i_{\alpha\beta}^{\beta})^* \omega_{\beta} \text{ for any } \alpha, \beta \right\}$$

Functoriality of $\Omega^{\bullet}(U_{\alpha}, \mathbb{R})$ implies that the definition of $\Omega^{\bullet}(M, \mathbb{R})$ is independent of any choices, is a DGA, and is functorial with respect to pull-back. To show that $\Omega^{\bullet}(M, \mathbb{R})$ is independent of the choice of the atlas, one chooses a common refinement and shows that $\Omega^{\bullet}(M, \mathbb{R})$ does not change under refinement. More intrinsically, one can define differential forms as sections of a sheaf or as sections of a vector bundle. We will return to this when we discuss sheaves.

6.2. EXAMPLE.

$$H_{dR}^0(M, \mathbb{R}) = \{f \in C^{\infty}(M) \mid df = 0\} = \{\text{locally constant functions on } M\}$$

In particular, $\dim H_{dR}^0(M, \mathbb{R})$ is the number of connected components of M .

§7. FREE, PROJECTIVE, AND INJECTIVE RESOLUTIONS. JAN 30.

Now we will use (co)homology to study a ring R rather than a space X .

7.1. LEMMA. Any R -module M admits a free resolution, i.e. an exact sequence

$$\dots \rightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \rightarrow 0,$$

where F_i is a free R -module for any i . If R is Noetherian and M is finitely generated, there exists a free resolution such that any F_i is finitely generated.

Proof. Choose generators $\{g_i\}_{i \in S}$ of M and define a surjective map

$$f_0 : F_0 = \bigoplus_{i \in S} R \rightarrow M, \quad e_i \mapsto g_i,$$

iterate to construct a surjection $f_1 : F_1 \rightarrow \text{Ker } f_0$, etc. Notice that $\text{Ker } f_0$ (and, by induction, $\text{Ker } f_i$ for any i) is finitely generated if R is Noetherian and M is finitely generated by g_i 's. \square

In more general Abelian categories (such as the category of complexes), it's not clear how to define a "free object". However, one can always define a "projective object", because the following definition makes sense in any Abelian category:

7.2. DEFINITION. An R -module P is called *projective*, if for any surjection of R -modules $A \xrightarrow{p} B \rightarrow 0$, and any map $P \xrightarrow{f} B$, there exists a map $P \xrightarrow{g} A$ such that $p \circ g = f$.

7.3. LEMMA. An R -module P is projective iff there exists an R -module P' such that $P \oplus P'$ is a free R -module. In particular, free R -modules are projective.

Proof. Proved in class. □

7.4. DEFINITION. An Abelian category A has *enough projective objects* if, for any object M , there exists a surjection $P \rightarrow M \rightarrow 0$, where P is a projective object.

Since free R -modules are projective, it is clear that $R\text{-mod}$ has enough projectives. It is also clear that any object in an Abelian category with enough projectives has a projective resolution, i.e. an exact sequence

$$\dots \rightarrow P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \rightarrow 0,$$

where any P_i is projective (just use the proof of Lemma 7.1).

Dually, we have

7.5. DEFINITION. An R -module I is called *injective*, if for any injection of R -modules $0 \rightarrow A \xrightarrow{i} B$, and any map $A \xrightarrow{f} I$, there exists a map $B \xrightarrow{g} I$ such that $g \circ i = f$. An *injective resolution* of an R -module M is an exact sequence

$$0 \rightarrow M \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1 \xrightarrow{f_2} I_2 \rightarrow \dots,$$

where any I_i is injective.

Similarly, we can define injective objects in any Abelian category. We say that an Abelian category has *enough injective objects* if any object is a subobject of an injective object. In a category like this, any object will have an injective resolution.

We are going to prove

7.6. THEOREM. $R\text{-mod}$ has enough injective objects, i.e. any R -module is a submodule of an injective module.

How do injective modules look like? First of all, we have

7.7. LEMMA. An injective R -module I is divisible, i.e. for any $x \in I$, and for any non-zero divisor $d \in R$, there exists $y \in I$ such that $x = dy$.

Proof. Apply definition of injectivity to homomorphisms $i : R \xrightarrow{\times d} R$ and $f : R \rightarrow I, 1 \mapsto x$. □

Moreover, divisibility characterizes injective Abelian groups:

7.8. LEMMA. An Abelian group I is an injective \mathbb{Z} -module iff I is divisible.

Proof. Consider any injection $i : A \hookrightarrow B$ of Abelian groups and a homomorphism $f : A \rightarrow I$. We want to extend it to a homomorphism $B \rightarrow I$, which we are going to denote by the same letter f . Applying Zorn's lemma², we reduce to the case $B = A + \mathbb{Z}x$. If the sum is direct, we can define $f(x)$ arbitrarily. If not, let d be the minimum positive integer such that

²Details: consider a partially ordered set (M, g) , where $A \subset M \subset B$ and $g : M \rightarrow I$ is a homomorphism that extends f . Here $(M, g) \leq (M', g')$ if $M \subset M'$ and $g'|_M = g$. This poset is non-empty (contains (A, f)), any increasing chain has an upper bound (just take a union of its subgroups), therefore by Zorn's lemma it contains a maximal element (M, g) . If $M = B$, we are done. Otherwise, let $x \in B \setminus M$. We are going to show that g can be extended to $M + \mathbb{Z}x$, which will give a contradiction.

$dx \in A$. By divisibility of I , we can find $u \in I$ such that $f(dx) = du$. We claim that $f(a + kx) := f(a) + ku$ is a well-defined homomorphism, where $a \in A$ and $k \in \mathbb{Z}$. Indeed, suppose

$$a + kx = a' + k'x.$$

Then

$$a - a' = (k' - k)x$$

and therefore

$$k' - k = md.$$

We have

$$f(a) + ku = f(a' + mdx) + ku = f(a') + mu + ku = f(a') + k'u.$$

□

7.9. COROLLARY. \mathbb{Q} and \mathbb{Q}/\mathbb{Z} are injective \mathbb{Z} -modules.

§8. EXACT FUNCTORS. FEB 1.

It is instructive to recast definitions of injective and projective modules (or more generally objects of an Abelian category) in terms of exactness of various functors. Recall that

8.1. DEFINITION. A covariant or contravariant functor $F : C_1 \rightarrow C_2$ between two Abelian categories is called *exact* (resp. *right exact*, resp. *left exact*) if it takes a short exact sequence $0 \rightarrow M \rightarrow N \rightarrow K \rightarrow 0$ to a sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, which is exact (resp. exact everywhere except at A , resp. exact everywhere except at C). Here we use a convention that arrows always go from left to right.

8.2. LEMMA. Fix an R -module X .

- (a) $\text{Hom}(X, \bullet)$ is a covariant left exact functor $R\text{-mod} \rightarrow R\text{-mod}$. It is exact if and only if X is projective.
- (b) $\text{Hom}(\bullet, X)$ is a contravariant left exact functor $R\text{-mod} \rightarrow R\text{-mod}$. It is exact if and only if X is injective.
- (c) $\bullet \otimes X$ is a covariant right exact functor $R\text{-mod} \rightarrow R\text{-mod}$. It is exact if and only if X is flat.

Proof. This is a reformulation of definitions. □

§9. **Ab** HAS ENOUGH INJECTIVES.

9.1. LEMMA. **Ab** has enough injectives, i.e. any Abelian group is a subgroup of an injective group.

Proof. Let M be an Abelian group. Define $M^\vee = \text{Hom}(M, \mathbb{Q}/b\mathbb{Z})$. We claim that the canonical map

$$\eta : M \rightarrow (M^\vee)^\vee, \quad m \mapsto [f \rightarrow f(m)]$$

is injective. Indeed, take $m \in M, m \neq 0$. Since \mathbb{Q}/\mathbb{Z} has elements of any finite order, there exists a homomorphism $f : \langle m \rangle \rightarrow \mathbb{Q}/\mathbb{Z}, f(m) \neq 0$. Since

\mathbb{Q}/\mathbb{Z} is an injective \mathbb{Z} -module, this homomorphism extends to a homomorphism $f : M \rightarrow \mathbb{Q}/\mathbb{Z}$. So η is injective. It remain to check that $(M^\vee)^\vee$ is a subgroup of a divisible group. Indeed, take a surjection

$$\bigoplus_{i \in I} \mathbb{Z} = F \twoheadrightarrow M^\vee,$$

where F is a free Abelian group. Dualization gives an injection

$$(M^\vee)^\vee \hookrightarrow F^\vee = \prod_{i \in I} (\mathbb{Q}/\mathbb{Z}),$$

and the last group is clearly injective. □

One lesson to take from this proof is that taking a double dual often improves the module.

§10. ADJOINT FUNCTORS.

This is some of the most amusing and useful part of category theory.

10.1. DEFINITION. Let $F : C \rightarrow C'$ and $G : C' \rightarrow C$ be covariant functors. We call F a *left adjoint* of G (and G a *right adjoint* of F), notation $F \dashv G$, if for any $X \in \mathbf{Ob}(C)$ and $Y \in \mathbf{Ob}(C')$ there exist bijections

$$\mathrm{Hom}_{C'}(FX, Y) \xrightarrow{\eta_{X,Y}} \mathrm{Hom}_C(X, GY),$$

which are natural in X and Y .

Recall that “natural” in category theory means “commutes with morphisms”. Namely, any morphism $X_1 \rightarrow X_2$ in C gives a diagram

$$\begin{array}{ccc} \mathrm{Hom}_{C'}(FX_1, Y) & \xrightarrow{\eta_{X_1,Y}} & \mathrm{Hom}_C(X_1, GY) \\ \uparrow & & \uparrow \\ \mathrm{Hom}_{C'}(FX_2, Y) & \xrightarrow{\eta_{X_2,Y}} & \mathrm{Hom}_C(X_2, GY) \end{array}$$

and any morphism $Y_1 \rightarrow Y_2$ in C' gives a diagram

$$\begin{array}{ccc} \mathrm{Hom}_{C'}(FX, Y_1) & \xrightarrow{\eta_{X,Y_1}} & \mathrm{Hom}_C(X, GY_1) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{C'}(FX, Y_2) & \xrightarrow{\eta_{X,Y_2}} & \mathrm{Hom}_C(X, GY_2) \end{array}$$

The requirement is that these diagrams are commutative.

This can also be rephrased in the language of natural transformations: we have two covariant functors

$$C^{op} \times C' \rightarrow \mathbf{Sets},$$

$$(X, Y) \mapsto \mathrm{Hom}_{C'}(FX, Y) \quad \text{and} \quad (X, Y) \mapsto \mathrm{Hom}_C(X, GY).$$

Then η gives a natural transformation between these functors.

For example, fix a ring homomorphism $\phi : S \rightarrow R$. Then we have the following functors

$$F : R\text{-mod} \rightarrow S\text{-mod}, \quad Y \mapsto Y, \quad (\text{restriction of scalars}), \quad (10.2)$$

with a multiplication $sm := \phi(s)m$ for $s \in S, m \in Y$, and

$$H : S\text{-mod} \rightarrow R\text{-mod}, \quad X \mapsto R \otimes_S X, \quad (\text{extension of scalars}).$$

We have

$$\text{Hom}_S(X, Y) \simeq \text{Hom}_R(R \otimes_S X, Y)$$

for any S -module X and any R -module Y , i.e. H is a left-adjoint of F .

§11. $R\text{-mod}$ HAS ENOUGH INJECTIVES. FEB 3.

This will be a fun application of adjunction. Consider again a ring homomorphism $S \rightarrow R$ and a restriction of scalars functor (10.2). What is its right adjoint? For any S -module Y , consider

$$F(Y) = \text{Hom}_S(R, Y).$$

R acts on $F(Y)$ by formula $(rf)(a) = f(ar)$. It is clear how to define F on maps $Y_1 \rightarrow Y_2$. So F is a functor. Let's construct a natural bijection

$$\text{Hom}_S(X, Y) \simeq \text{Hom}_R(X, \text{Hom}_S(R, Y))$$

for any R -module X and S -module Y . In one direction, we send $\psi : X \rightarrow Y$ to $f(x)(a) = \psi(ax)$. In the other direction, we send $f : X \rightarrow \text{Hom}_S(R, Y)$ to $\psi(x) = f(x)(1)$. It is easy to see that these maps are inverses of each other.

11.1. LEMMA. *If Y is an injective S -module then $F(Y) = \text{Hom}_S(R, Y)$ is an injective R -module.*

Proof. We have to show that $\text{Hom}_R(\bullet, F(Y))$ is an exact functor. Notice that G is an exact functor by obvious reasons. Therefore, by adjunction, it suffices to check that $\text{Hom}_S(\bullet, Y)$ is an exact functor. Since Y is injective, this is indeed the case. \square

Now we can finish the proof that $R\text{-mod}$ has enough injectives. Viewing an R -module M as an Abelian group, we have an injective homomorphism of Abelian groups

$$M \hookrightarrow T,$$

where T is an injective \mathbb{Z} -module, i.e. a divisible group. By adjunction, this gives a homomorphism of R -modules

$$M \rightarrow \text{Hom}_{\mathbb{Z}}(R, T)$$

which is clearly injective. By the previous Lemma, $\text{Hom}_{\mathbb{Z}}(R, T)$ is an injective R -module.

§12. QUASI-ISOMORPHISM AND HOMOTOPY. FEB 6.

Intuitively, one is tempted to think that complexes are too dependent on auxiliary choices to be useful, and one should work with their cohomology instead. After all, it's singular homology and not the singular chain complex that gives invariants of the topological space. However, this intuition is wrong. For example, the most useful tool in homological algebra is a connecting homomorphism, which is defined only on the level of complexes. A better thing to do is to consider a category of all complexes but to modify morphisms to better reflect cohomological information.

12.1. DEFINITION. A map of (co)chain complexes is called a *quasi-isomorphism* if it induces an isomorphism of (co)homology.

For example, at some point we will prove

12.2. THEOREM. Let X be a simplicial complex with geometric realization $|X|$. The inclusion of simplicial chains into singular chains

$$C_{\bullet}^{simp}(X, \mathbb{Z}) \hookrightarrow C_{\bullet}^{sing}(|X|, \mathbb{Z})$$

is a quasi-isomorphism.

Quasi-isomorphisms should be viewed as almost invertible maps.

12.3. DEFINITION. Let $f, g : C_{\bullet} \rightarrow C'_{\bullet}$ be maps of chain complexes (of R -modules). They are called homotopic if there exist maps $p_n : C_n \rightarrow C'_{n+1}$ such that

$$d_{n+1} \circ p_n + p_{n+1} \circ d_n = f_n - g_n$$

for any n . One has to modify definitions accordingly for cochain complexes.

12.4. LEMMA. Homotopic maps of complexes induce identical maps on homology.

Proof. It suffices to show that $d_{n+1} \circ p_n + p_{n+1} \circ d_n$ sends any cycle α to a boundary. But this is clear:

$$d_{n+1} \circ p_n(\alpha) + p_{n+1} \circ d_n(\alpha) = d_{n+1}(p_n(\alpha)).$$

□

A useful corollary is a method of checking that a complex C_{\bullet} is acyclic: it suffices to construct a homotopy between the identity map and the zero map.

12.5. DEFINITION. Two complexes C, C' are called *homotopy equivalent* if there exists maps of complexes $\alpha : C \rightarrow C', \beta : C' \rightarrow C$ such that $\alpha\beta$ is homotopic to $\text{Id}_{C'}$ and $\beta\alpha$ is homotopic to Id_C .

By the previous lemma, in this case both f and g are quasi-isomorphisms. As an example, let's show

12.6. THEOREM. Any two projective (resp. injective) resolutions of an R -module M are homotopy equivalent.

Proof. We will show even more. Let

$$\dots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

and

$$\dots \rightarrow P'_2 \rightarrow P'_1 \rightarrow P'_0 \rightarrow M \rightarrow 0$$

be two projective resolutions. It suffices to prove the following:

- (a) The identity map Id_M extends to map of complexes $P_{\bullet} \rightarrow P'_{\bullet}$.
- (b) Any two extensions $f, g : P_{\bullet} \rightarrow P'_{\bullet}$ of Id_M are homotopic.

Both items were checked in class. To show homotopy equivalence of P_{\bullet} and P'_{\bullet} , use (a) to construct maps $\alpha : P_{\bullet} \rightarrow P'_{\bullet}$ and $\beta : P'_{\bullet} \rightarrow P_{\bullet}$. Then use (b) to construct homotopy of $\alpha\beta$ with $\text{Id}_{P'}$ and $\beta\alpha$ with Id_P . □

§13. HOMOTOPY INVARIANCE OF SINGULAR HOMOLOGY. FEB 8

Recall that continuous maps of topological spaces $f, g : X \rightarrow Y$ are called *homotopic* if there exists a continuous map $F : X \times [0, 1] \rightarrow Y$ such that $f(x) = F(x, 0)$ and $g(x) = F(x, 1)$. The following theorem explains origins of the notion of homotopic complexes.

13.1. THEOREM. *Homotopic maps of topological spaces $f, g : X \rightarrow Y$ induce homotopic maps of singular chain complexes $f_*, g_* : C_\bullet^{sing}(X, \mathbb{Z}) \rightarrow C_\bullet^{sing}(Y, \mathbb{Z})$.*

13.2. COROLLARY. *Homotopic maps of topological spaces $f, g : X \rightarrow Y$ induce equal maps of singular homology $f_*, g_* : H_\bullet^{sing}(X, \mathbb{Z}) \rightarrow H_\bullet^{sing}(Y, \mathbb{Z})$.*

13.3. COROLLARY. *Homotopy equivalent³ topological spaces X and Y have isomorphic singular homology.*

13.4. COROLLARY. *If X is a contractible topological space then*

$$H_i^{sing}(X, \mathbb{Z}) = H_i^{sing}(pt, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i > 0 \end{cases}$$

To prove the Theorem, we are going to construct homotopy homomorphisms $p_n : C_n(X, \mathbb{Z}) \rightarrow C_{n+1}(Y, \mathbb{Z})$, also known as *prism maps*. Recall that the required condition is that

$$\partial p_n = g_* - f_* - p_{n-1} \partial,$$

where ∂ is the chain differential. We are going to define p_n on simplices and extend by linearity. Take $\sigma \in C_n(X, \mathbb{Z})$, i.e. a continuous map $\sigma : \Delta_n \rightarrow X$. We have a continuous map

$$H : \Delta_n \times [0, 1] \xrightarrow{\sigma \times \text{Id}} X \times [0, 1] \xrightarrow{F} Y.$$

Notice that

$$H|_{\Delta_n \times \{0\}} = f_*(\sigma), \quad H|_{\Delta_n \times \{1\}} = g_*(\sigma).$$

We label the “bottom” vertices of the prism $\Delta_n \times [0, 1]$ by $0, 1, \dots, n$, and the “top” vertices by $\bar{0}, \dots, \bar{n}$. Then we define

$$p_n(\sigma) = \sum_{i=0}^n (-1)^i H \circ [0, \dots, i, \bar{i}, \dots, \bar{n}],$$

where $[0, \dots, i, \bar{i}, \dots, \bar{n}]$ is an affine-linear map from the Δ_{n+1} simplex to the prism that sends vertices $0, \dots, n+1$ of Δ_{n+1} to the vertices $0, \dots, i, \bar{i}, \dots, \bar{n}$ of the prism. A picture would be very enlightening here. Now we compute

$$\begin{aligned} \partial p_n(\sigma) &= \left(\sum_{j < i} (-1)^{i+j} H \circ [0, \dots, \hat{j}, \dots, i, \bar{i}, \dots, \bar{n}] + \sum_{j > i} (-1)^{i+j+1} H \circ [0, \dots, i, \bar{i}, \dots, \hat{j}, \dots, \bar{n}] \right) + \\ &\quad \sum_i H \circ [0, \dots, \hat{i}, \bar{i}, \dots, \bar{n}] - \sum_i H \circ [0, \dots, i, \hat{i}, \dots, \bar{n}] = \\ &\quad p_{n-1} \partial(\sigma) + H \circ [\hat{0}, \bar{0}, \dots, \bar{n}] - H \circ [0, \dots, n, \hat{n}] = \\ &\quad p_{n-1} \partial(\sigma) + g_*(\sigma) - f_*(\sigma). \end{aligned}$$

³Recall that this means existence of maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that fg is homotopic to Id_Y and gf is homotopic to Id_X .

§14. HOMOTOPY INVARIANCE OF DE RHAM COHOMOLOGY.

In the same spirit, let's prove

14.1. THEOREM. *Smoothly homotopic smooth maps $f, g : X \rightarrow Y$ of smooth manifolds. induce homotopic maps of de Rham complexes*

$$f^*, g^* : \Omega^\bullet(Y, \mathbb{R}) \rightarrow \Omega^\bullet(X, \mathbb{R}).$$

14.2. COROLLARY. *Smoothly homotopic smooth maps of smooth manifolds $f, g : X \rightarrow Y$ induce equal maps of de Rham cohomology $f^*, g^* : H_{dR}^\bullet(Y, \mathbb{R}) \rightarrow H_{dR}^\bullet(X, \mathbb{R})$.*

14.3. COROLLARY. *Smoothly homotopy equivalent smooth manifolds X and Y have isomorphic de Rham cohomology.*

14.4. COROLLARY (Poincare Lemma). *If X is a smoothly contractible smooth manifold then*

$$H_{dR}^i(X, \mathbb{R}) = H_{dR}^i(pt, \mathbb{R}) = \begin{cases} \mathbb{R} & i = 0 \\ 0 & i > 0 \end{cases}$$

To prove the Theorem, we construct homotopy homomorphisms

$$p_k : \Omega^k(Y) \rightarrow \Omega^{k-1}(X),$$

as follows. Take $\omega \in \Omega^k(Y)$. We are given a smooth homotopy

$$F : X \times [0, 1] \rightarrow Y.$$

Let t be a coordinate along $[0, 1]$, and let x_1, \dots, x_n be local coordinates on X . Then we have

$$F^*\omega = \sum_{|I|=k} a_I(x, t) dx_I + \sum_{|J|=k-1} b_J(x, t) dt \wedge dx_J,$$

where we use a common notation $dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_k}$. We define

$$p_k(\omega) = \sum_{|J|=k-1} \left(\int_0^1 b_J(x, t) dt \right) dx_J.$$

It is easy to see that this definition is independent on the choice of local coordinates. Now we compute

$$\begin{aligned} d(p_k(\omega)) &= \sum_{|J|=k-1} \frac{\partial \left(\int_0^1 b_J(x, t) dt \right)}{\partial x_j} dx_j \wedge dx_J \\ &= \sum_{|J|=k-1} \left(\int_0^1 \frac{\partial b_J(x, t)}{\partial x_j} dt \right) dx_j \wedge dx_J, \\ F^*(d\omega) = d(F^*(\omega)) &= \sum_{|I|=k} \frac{\partial a_I(x, t)}{\partial t} dt \wedge dx_I + (\text{terms without } dt) \\ &+ \sum_{|J|=k-1} \frac{\partial b_J(x, t)}{\partial x_j} dx_j \wedge dt \wedge dx_J, \end{aligned}$$

and therefore

$$p_{n+1}(d(\omega)) = \sum_{|I|=k} \left(\int_0^1 \frac{\partial a_I(x, t)}{\partial t} dt \right) dx_I \\ - \sum_{\substack{j \\ |J|=k-1}} \left(\int_0^1 \frac{\partial b_J(x, t)}{\partial x_j} \right) dx_j \wedge dx_J =$$

(Newton–Leibniz)

$$\sum_{|I|=k} a(x, 1) dx_I - \sum_{|I|=k} a(x, 0) dx_I - p_k(\omega) = \\ g^*(\omega) - f^*(\omega) - p_k(\omega).$$

Indeed, $F(x, 1) = g(x)$ implies $g^*(\omega) = F^*(\omega)|_{X \times \{1\}}$, which shows that $g^*(\omega) = \sum_{|I|=k} a(x, 1) dx_I$ and similarly for $f^*(\omega)$.

§15. DOLBEAUT COMPLEX AND $\bar{\partial}$ -POINCARÉ LEMMA. FEB 10.

Constructing algebraic homotopy maps is the most straightforward way of showing exactness of the complex, but sometimes complexes are exact by mysterious reasons. Prime examples are Dolbeaut complex and Koszul complex.

Let M be a complex manifold. This means that M has open subsets of \mathbb{C}^n as charts and transition functions are holomorphic. As a real $2n$ -dimensional smooth manifold, M has a de Rham complex, and we consider its complexified version

$$\Omega^k(M, \mathbb{C}) = \Omega^k(M, \mathbb{R}) \otimes \mathbb{C}.$$

This vector space parametrizes complex-valued C^∞ differential k -forms on M . For example, locally, we have complex coordinates $z_p = x_p + ix_{n+p}$, $p = 1, \dots, n$, and their differentials

$$dz_p = dx_p + idx_{n+p}, \quad d\bar{z}_p = dx_p - idx_{n+p}.$$

For any C^∞ -function f , we can write its differential as

$$df = \sum_{p=1}^n \frac{\partial f}{\partial x_p} dx_p + \sum_{p=1}^n \frac{\partial f}{\partial x_{p+n}} dx_{p+n} = \\ \sum_{p=1}^n \frac{\partial f}{\partial x_p} \frac{1}{2} (dz_p + d\bar{z}_p) + \sum_{p=1}^n \frac{\partial f}{\partial x_{p+n}} \frac{1}{2i} (dz_p - d\bar{z}_p) = \\ \sum_{p=1}^n \frac{1}{2} \left(\frac{\partial f}{\partial x_p} - i \frac{\partial f}{\partial x_{p+n}} \right) dz_p + \sum_{p=1}^n \frac{1}{2} \left(\frac{\partial f}{\partial x_p} + i \frac{\partial f}{\partial x_{p+n}} \right) d\bar{z}_p = \\ \sum_{p=1}^n \frac{\partial f}{\partial z_p} dz_p + \sum_{p=1}^n \frac{\partial f}{\partial \bar{z}_p} d\bar{z}_p = \partial f + \bar{\partial} f.$$

The last line is just the definition of $\frac{\partial f}{\partial z_p}$, $\frac{\partial f}{\partial \bar{z}_p}$, ∂f , and $\bar{\partial} f$ (“Cauchy–Riemann equations”). Recall that f is called holomorphic if $\bar{\partial} f = 0$.

We extend $d = \partial + \bar{\partial}$ to the full de Rham complex. First, we have a decomposition

$$\Omega^k(M, \mathbb{C}) = \bigoplus_{p+q=k} A^{p,q}, \quad (15.1)$$

where $A^{p,q}$ parametrizes k -forms that locally look like

$$\omega = \sum_{\substack{|I|=p \\ |J|=q}} g_{I,J}(z_1, \dots, z_n) dz_I \wedge d\bar{z}_J.$$

Notice that a (p, q) -form pulls back to a (p, q) -form under a holomorphic map, and in particular the decomposition (15.1) is intrinsic, i.e. does not depend on local holomorphic coordinates. Indeed, if a holomorphic map f is given by $z_i = f_i(w_1, \dots, w_m)$, where each f_i is holomorphic, then

$$\begin{aligned} f^*\omega &= \sum_{\substack{|I|=p \\ |J|=q}} g_{I,J}(f_1(w), \dots, f_n(w)) df_{i_1} \wedge \dots \wedge df_{i_p} \wedge d\bar{f}_{j_1} \wedge \dots \wedge d\bar{f}_{j_q} = \\ & \sum_{\substack{|I|=p \\ |J|=q}} g_{I,J}(f_1(w), \dots, f_n(w)) \partial f_{i_1} \wedge \dots \wedge \partial f_{i_p} \wedge \bar{\partial} \bar{f}_{j_1} \wedge \dots \wedge \bar{\partial} \bar{f}_{j_q} \in A^{p,q}. \end{aligned}$$

It is also clear that if $\omega \in A^{p,q}$ then $d\omega = \partial\omega + \bar{\partial}\omega$, where

$$\partial\omega = \sum_{\substack{|I|=p \\ |J|=q}} (\partial g_{I,J}) dz_I \wedge d\bar{z}_J \in A^{p+1,q}$$

and

$$\bar{\partial}\omega = \sum_{\substack{|I|=p \\ |J|=q}} (\bar{\partial} g_{I,J}) dz_I \wedge d\bar{z}_J \in A^{p,q+1}.$$

So we have a decomposition

$$d = \partial + \bar{\partial}, \quad \partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$$

and for each p , we have a *Dolbeault complex* $(A^{p,\bullet}, \bar{\partial})$. This is an instance of the following general definition:

15.2. DEFINITION. A *double complex* is a collection of objects $C^{p,q}$ of an Abelian category along with maps

$$d_1 : C^{p,q} \rightarrow C^{p+1,q}, \quad d_2 : C^{p,q} \rightarrow C^{p,q+1}$$

such that

$$d_1^2 = d_2^2 = d_1d_2 + d_2d_1 = 0.$$

So we have a vertical complex for each p , a horizontal complex for each q . We also define a total complex $(\mathbf{tot}^\bullet(C), d)$ of a double complex as follows:

$$\mathbf{tot}^k(C) = \bigoplus_{p+q=k} C^{p,q}, \quad d = d_1 + d_2.$$

It is clear that $d^2 = 0$.

Finally, we can introduce *Dolbeault cohomology* of a complex manifold:

$$H^{p,q}(M, \mathbb{C}) = \frac{\text{Ker}[A^{p,q} \xrightarrow{\bar{\partial}} A^{p,q+1}]}{\text{Im}[A^{p,q-1} \xrightarrow{\bar{\partial}} A^{p,q}]}.$$

By analogy with Poincaré lemma, we expect this cohomology to vanish if M is a neighborhood of a point. In fact, we have

15.3. THEOREM ($\bar{\partial}$ -Poincaré Lemma). *Let*

$$\Delta = \{(z_1, \dots, z_n) \mid |z_i| < r_i\} \subset \mathbb{C}^n$$

be a polydisk. Then $H^{p,q}(\Delta, \mathbb{C}) = 0$ for $q > 0$.

Notice that the Dolbeault complex $A^{p,\bullet}(\Delta, \mathbb{C})$ is just a direct sum of $\binom{n}{p}$ copies of $A^{0,\bullet}(\Delta, \mathbb{C})$ (one copy for each dz_I , $|I| = p$). So it suffices to prove that $H^{0,q}(\Delta, \mathbb{C}) = 0$ for $q > 0$, i.e. that we have an exact sequence

$$0 \rightarrow \mathcal{O}_{hol}(\Delta) \rightarrow A^{0,0} \xrightarrow{\bar{\partial}} A^{0,1} \xrightarrow{\bar{\partial}} \dots$$

We take a $\bar{\partial}$ -closed form $w \in A^{0,q}$ and pretty much construct by hand a form $\tau \in A^{0,q-1}$ such that $\bar{\partial}\tau = w$. The proof boils down to the Cauchy integral formula from complex analysis after decorating it with some bells and whistles:

15.4. THEOREM (Cauchy). *Let $\Delta = \{|z| < 1\} \subset \mathbb{C}$ be the (open) unit disk. Let $f(z)$ be a complex-valued C^∞ function on $\bar{\Delta}$. Then, for any $z \in \Delta$,*

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Delta} \frac{f(w) dw}{w - z} + \frac{1}{2\pi i} \int_{\Delta} \frac{\partial f}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w - z}$$

This formula is usually applied when f is a holomorphic function, in which case the second term vanishes. However, in our application it is the first term that trivially vanishes and all the action is happening with the second term! We followed very closely exposition in Griffiths–Harris, pages 5 and 25.

§16. KOSZUL COMPLEX. FEB 13.

This is a little gadget from commutative algebra that packs an incredible amount of information. Warning: there are different conventions about signs, indexing, etc. in the Koszul complex. They don't change the substance though.

Setup: a Noetherian commutative ring R , a bunch of elements $x_1, \dots, x_n \in R$, and, for finer applications, a finitely generated R -module M .

Case $n = 1$. Here we simply have $x \in R$, and the Koszul complex is a 2-term complex

$$K(x) : \quad 0 \rightarrow R \xrightarrow{x} R \rightarrow 0,$$

The complex is assumed to start in degree 0, with differentials increasing the degree. So we have

$$H^0 = \text{Ann}(x), \quad H^1 = R/(x).$$

Therefore,

$$H^1 \neq 0 \quad \Leftrightarrow \quad R/(x) \neq 0,$$

$$H^0 = 0 \iff x \text{ is not a zero-divisor.}$$

This can be upgraded if we have a module. We define

$$\begin{aligned} K(x; M) = K(x) \otimes M : \quad & 0 \rightarrow M \xrightarrow{x} M \rightarrow 0, \\ H^1 = M/xM, \quad & H^1 \neq 0 \iff M/xM \neq 0, \\ H^0 = (M : x) \quad & H^0 = 0 \iff x \text{ is not a zero-divisor in } M. \end{aligned}$$

Case n = 2. Now the fun really starts. We fix $x, y \in R$, and define

$$K(x, y) : \quad 0 \rightarrow R \xrightarrow{\begin{bmatrix} x \\ y \end{bmatrix}} R \oplus R \xrightarrow{\begin{bmatrix} -y & x \end{bmatrix}} R \rightarrow 0,$$

and $K(x, y; M) = K(x, y) \otimes M$:

$$0 \rightarrow M \xrightarrow{\begin{bmatrix} x \\ y \end{bmatrix}} M \oplus M \xrightarrow{\begin{bmatrix} -y & x \end{bmatrix}} M \rightarrow 0.$$

The indexing starts with 0 and ends with 2. It is clear that we have

$$H^2(K(x, y)) = R/(x, y), \quad H^2(K(x, y; M)) = M/(x, y)M.$$

So $H^2 \neq 0$ is equivalent to $M \neq (x, y)M$. How about H^0 ? Following definitions, we see that

$$H^0(K(x, y)) = \text{Ann}(x) \cap \text{Ann}(y) = \text{Ann}(x, y),$$

and $H^0(K(x, y; M))$ is the set of elements of M annihilated by (x, y) . So, $H^0(K(x, y; M)) = 0$ means that every element of M is not annihilated by some element of (x, y) . For example, $H^0(K(x, y)) = 0$ means that every element of R is not annihilated either by x or by y . Here is the first non-trivial fact

16.1. LEMMA. $H^0(K(x, y)) = 0$ if and only if (x, y) contains a non-zero-divisor. $H^0(K(x, y; M)) = 0$ if and only if $\exists a \in (x, y)$, which is not a zero-divisor on M .

This will be explained in the next section.

§17. ASSOCIATED PRIMES

17.1. DEFINITION. A prime ideal $\mathfrak{p} \subset R$ is called *associated* with an R -module M if $\mathfrak{p} = \text{Ann}(m)$ for some $m \in M$. The set of associated primes is denoted by

$$\text{Ass}(M) \subset \text{Spec } R.$$

For example, take $R = \mathbb{Z}$, $M = \mathbb{Z}/n\mathbb{Z}$. Then associated primes are prime divisors of n .

17.2. DEFINITION. Minimal (by inclusion) prime ideals of R containing $\text{Ann}(M)$ are called *minimal* prime ideals of M . Let $\text{Min}(M) \subset \text{Spec } R$ be the set of minimal prime ideals of M .

17.3. THEOREM. Assume R is Noetherian, M is finitely generated. Then

- $\text{Min}(M) \subset \text{Ass}(M)$ are finite non-empty sets.
- $\bigcup_{\mathfrak{p} \in \text{Ass}(M)} \mathfrak{p}$ is equal to the set of elements of R which are zero-divisors in M .

Here is a standard corollary:

17.4. COROLLARY. *Under the same assumptions, suppose that every element of an ideal $I \subset R$ annihilates some element of M . Then there exists a non-zero $m \in M$ such that $I \subset \text{Ann}(m)$.*

Proof. Since I is contained in the set of elements of R which are zero-divisors in M , I belongs to the union of associated primes. By prime avoidance, I is contained in one of associated primes $\mathfrak{p} = \text{Ann}(m)$. \square

A contrapositive is even more useful:

17.5. COROLLARY. *The following statements are equivalent:*

- For any non-zero element $m \in M$, $I \not\subset \text{Ann}(m)$.
- There exists $a \in I$ which is not a zero-divisor on M .

Lemma 16.1 clearly follows from this corollary.

§18. REGULAR SEQUENCES. FEB 15.

Now let's look at $H^1(K(x, y))$. By definition, we have

$$H^1 = \frac{\{(a, b) \in R \oplus R \mid -ya + xb = 0\}}{\{(xc, yc) \in R \oplus R \mid c \in R\}}.$$

This means that $a \in ((x) : y)$. For simplicity, let's assume x is not a zero-divisor. Then b is uniquely determined by a , and we have

$$H^1 = \frac{(x) : y}{(x)}.$$

So $H^1 = 0$ if and only if y is not a zero-divisor modulo (x) .

18.1. DEFINITION. A sequence $x_1, \dots, x_n \in R$ is called a *regular sequence* if $R/(x_1, \dots, x_n) \neq 0$, x_1 is not a zero-divisor, and x_i is not a zero-divisor modulo (x_1, \dots, x_{i-1}) for any i .

18.2. DEFINITION. A sequence $x_1, \dots, x_n \in R$ is called an *M -sequence* if $M/(x_1, \dots, x_n)M \neq 0$, x_1 is not a zero-divisor on M , and x_i is not a zero-divisor in $M/(x_1, \dots, x_{i-1})M$ for any i .

In this language, we proved that if $\{x, y\}$ is a regular sequence then

$$H^i(K(x, y)) = \begin{cases} 0 & i < 2 \\ R/(x, y) \neq 0 & i = 2 \end{cases}.$$

A similar statement holds if x, y is an M -sequence. Is the converse true?

By definition, $K(x, y)$ is the total complex of a double complex

$$\begin{array}{ccc} R & \xrightarrow{x} & R \\ y \downarrow & & \downarrow y \\ R & \xrightarrow{-x} & R \end{array} \quad (1)$$

Notice that the bottom row is its subcomplex and the top row is a quotient complex. This is not quite right: we have to shift indices in the bottom row so that the first group has degree 1 and not 0. So more precisely, we have an exact sequence of complexes

$$0 \rightarrow K(x)[-1] \rightarrow K(x, y) \rightarrow K(x) \rightarrow 0.$$

Here we use the following notation:

18.3. NOTATION. Let C^\bullet be a complex. Then $C[k]^\bullet$ has the same cochains as C^\bullet but with shifted grading $C[k]^i = C^{k+i}$. It is convenient to set the differential in $C[k]$ to be the differential in C multiplied by $(-1)^k$ (notice that this has no effect on cocycles, coboundaries, and cohomology).

The long exact sequence in cohomology then gives

$$H^0(K(x, y)) \rightarrow H^0(K(x)) \rightarrow H^1(K(x))[-1] \rightarrow H^1(K(x, y)) \rightarrow H^1(K(x)) \rightarrow \dots$$

Under our assumptions we get

$$0 \rightarrow H^0(K(x)) \xrightarrow{\delta} H^1(K(x))[-1] \rightarrow 0,$$

i.e.

$$0 \rightarrow H^0(K(x)) \xrightarrow{\delta} H^0(K(x)) \rightarrow 0,$$

where δ is a connecting homomorphism. To compute it, get back to the double complex (1) picture of the Koszul complex. Start with a cocycle $z \in H^0(K(x))$ (upper-left corner). Then $xz = 0$. Lift it to $K^0(x, y)$, i.e. view z as an element of $K^0(x, y)$. Applying a differential $\begin{bmatrix} x \\ y \end{bmatrix}$ gives $\begin{bmatrix} 0 \\ yz \end{bmatrix} \in K^1(x, y)$. This has to be an element of a subcomplex, and it is, yz is in the left bottom corner. So $\delta(z) = yz$, i.e. a connecting homomorphism $H^0(K(x)) \rightarrow H^0(K(x))$ is multiplication by y ! The exact sequence then gives $yH^0(K(x)) = H^0(K(x))$. To show that x, y is a regular sequence we would need to show that x is not a zero-divisor, i.e. that $H^0(K(x)) = 0$. Notice that if R is a local ring and x, y belong to the maximal ideal, this is implied by Nakayama's Lemma! So we see that

18.4. COROLLARY. *If R is a local Noetherian ring then x, y is a regular sequence if and only if*

$$H^i(K(x, y)) = \begin{cases} 0 & i < 2 \\ \neq 0 & i = 2 \end{cases}.$$

If M is a finitely generated module then x, y is an M -sequence if and only if

$$H^i(K(x, y; M)) = \begin{cases} 0 & i < 2 \\ \neq 0 & i = 2 \end{cases}.$$

This already has very interesting consequences:

18.5. COROLLARY. *If R is a local Noetherian ring and x_1, \dots, x_n is a regular sequence then $x_{\sigma(1)}, \dots, x_{\sigma(n)}$ is a regular sequence for any permutation σ .*

Proof. It suffices to prove this when $\sigma = (i, i + 1)$ is a transposition of adjacent indices. Working in $R' = R/(x_1, \dots, x_{i-1})$, we then reduce to the case $n = 2$. If x, y is a regular sequence then $H^i(K(x, y))$ vanishes if and only if $i < 2$. The preceding analysis of the Koszul complex works just as fine if we interchange x and y (e.g. the double complex (1) will be reflected along the diagonal). So, by the previous corollary, y, x is a regular sequence as well. \square

18.6. EXAMPLE. Here is an example that shows that this corollary can fail if R is not local. Take $R = k[x, y, z]/(x - 1)z$. Then $\{x, (x - 1)y\}$ is a regular sequence but $\{(x - 1)y, x\}$ is not.

It follows that if R is not local then vanishing of Koszul complex cohomology groups does not quite detect regularity of a sequence (because the Koszul complex clearly does not care about the order of elements in the sequence). In fact, it detects the maximal length of a regular sequence in the ideal (x_1, \dots, x_n) .

18.7. DEFINITION. Let $I \subset R$ be an ideal such that $IM \neq M$. Then $\text{depth}(I, M)$ is, by definition, the maximal length of an M -sequence in I . In particular, $\text{depth } I$ is the maximal length of a regular sequence in I . We are also going to say that an M -sequence is maximal if one can not add elements to it (on the right).

So the depth is the maximal length of a maximal M -sequence. In fact, we are going to see that all maximal M -sequences in I have the same finite length (under our running assumptions that R is Noetherian and M is finitely generated).

But first let's define the Koszul complex in general.

18.8. DEFINITION. Let N be an R -module and let $x \in N$. The Koszul complex $K(x, N)$ is defined as follows:

$$0 \rightarrow \Lambda^0 N \xrightarrow{x \wedge \bullet} \Lambda^1 N \xrightarrow{x \wedge \bullet} \Lambda^2 N \xrightarrow{x \wedge \bullet} \Lambda^3 N \xrightarrow{x \wedge \bullet} \dots,$$

and the degree of the $\Lambda^0 N \simeq R$ component is 0.

Some remarks:

- $K(x, N)$ is a DGA.
- Functoriality: any R -linear map $f : N \rightarrow M$ induces a map of complexes $K(x, N) \rightarrow K(f(x), M)$.
- Another functoriality: for any $r \in R$, a multiplication by r map $\Lambda^\bullet N \rightarrow \Lambda^\bullet N$ is a map of complexes.

18.9. DEFINITION. For any sequence $x = (x_1, \dots, x_n) \in R^n$, we define a Koszul complex $K(x_1, \dots, x_n)$ as $K(x, R^n)$. For any R -module M , we define $K(x_1, \dots, x_n; M)$ as $K(x_1, \dots, x_n) \otimes_R M$ with the differential $d \otimes \text{Id}$.

More explicitly, $K(x_1, \dots, x_n)$ looks like

$$0 \rightarrow R \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \xrightarrow{\quad} R^n \xrightarrow{(x_1 e_1 + \dots + x_n e_n) \wedge \bullet} \Lambda^2 R^n \xrightarrow{(x_1 e_1 + \dots + x_n e_n) \wedge \bullet} \Lambda^3 R^n \rightarrow \dots \\ \dots \rightarrow \Lambda^{n-1} R^n \xrightarrow{(x_1 e_1 + \dots + x_n e_n) \wedge \bullet} \Lambda^n R^n \rightarrow 0$$

Notice that the last map is essentially

$$R^n \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} R,$$

where we use the basis $\{e_1 \wedge \dots \wedge \hat{e}_i \wedge \dots \wedge e_n\}$ of $\Lambda^{n-1} R^n$ to identify the latter with R^n . So

$$H^n K(x_1, \dots, x_n) = R/(x_1, \dots, x_n), \quad H^n K(x_1, \dots, x_n; M) = M/(x_1, \dots, x_n)M.$$

18.10. THEOREM. Let R be a Noetherian ring, M a finitely generated R -module, and $I = (x_1, \dots, x_n)$. Suppose $M \neq IM$. Define r as follows:

$$H^j(x_1, \dots, x_n; M) = \begin{cases} 0 & j < r \\ \neq 0 & j = r \end{cases}$$

Then every maximal M -sequence in I has length r and $\text{depth}(I, M) = r$.

18.11. COROLLARY. $\text{depth}((x_1, \dots, x_n), M) \leq n$.

18.12. COROLLARY. If x_1, \dots, x_n is a regular sequence then

$$H^j(x_1, \dots, x_n) = \begin{cases} 0 & j < n \\ \neq 0 & j = n \end{cases}.$$

In particular, we have an exact sequence

$$0 \rightarrow R \rightarrow R^n \rightarrow \Lambda^2 R^n \rightarrow \Lambda^3 R^n \rightarrow \dots \rightarrow \Lambda^n R^n \rightarrow R/(x_1, \dots, x_n) \rightarrow 0 \tag{18.13}$$

18.14. EXAMPLE. Take $x_1, \dots, x_n \in R = k[x_1, \dots, x_n]$. This is clearly a regular sequence and $R/(x_1, \dots, x_n) \simeq k$. So (18.13) gives a free resolution (=resolution by free modules) of k viewed as an R -module.

In the local setting we can say more.

18.15. THEOREM. In addition to assumptions of the previous theorem, let's also assume that R is local. Then $H^k(x_1, \dots, x_n; M) = 0$ implies that $H^j(x_1, \dots, x_n; M) = 0$ for any $j < k$. Moreover, $H^{n-1}(x_1, \dots, x_n; M) = 0$ implies that x_1, \dots, x_n is an M -sequence.

From now on we are going to focus on the case when $M = R$, and so M -sequences = regular sequences. In fact, all key features of the argument were already covered in our discussion of the $n = 2$ case. We just have to generalize an inductive description of the Koszul complex.

§19. MAPPING CONE. FEB 17.

19.1. DEFINITION. Let $\alpha : (F^\bullet, d_F) \rightarrow (G^\bullet, d_G)$ be a map of cochain complexes. Its *mapping cone* $M(\alpha)$ is a cochain complex with

$$M(\alpha)^i = F^{i+1} \oplus G^i$$

and the differential given by a matrix

$$\begin{bmatrix} -d_F & 0 \\ \alpha & d_G \end{bmatrix}. \tag{19.2}$$

Notice that G^\bullet is a subcomplex of $M(\alpha)^\bullet$, and the quotient complex is $F^\bullet[1]$ (recall that $[1]$ means shifting degrees by 1 and changing the sign of the differential. So we have a short exact sequence of complexes

$$0 \rightarrow G \rightarrow M(\alpha) \rightarrow F[1] \rightarrow 0$$

and the induced long exact sequence of cohomology

$$\dots \rightarrow H^k(G) \rightarrow H^k(M(\alpha)) \rightarrow H^{k+1}(F) \xrightarrow{\delta} H^{k+1}(G) \rightarrow \dots,$$

where δ is the connecting homomorphism.

19.3. LEMMA. $\delta = H(\alpha)$ (a map in cohomology induced by α).

Proof. This is a simple diagram chase (done in class) using the definition of the connecting homomorphism. \square

§20. INDUCTIVE DESCRIPTION OF THE KOSZUL COMPLEX. FEB 22.

We apply the mapping cone construction to the Koszul complex:

20.1. LEMMA. Let $\alpha : K(x_1, \dots, x_{n-1}) \rightarrow K(x_1, \dots, x_{n-1})$ be the map given by multiplication by $-x_n$. Then the mapping cone $M(\alpha)$ is isomorphic to the Koszul complex $K(x_1, \dots, x_n)[1]$ (shifted by 1). In particular, we have a long exact sequence

$$\begin{aligned} \dots \rightarrow H^{i-1}K(x_1, \dots, x_{n-1}) &\rightarrow H^iK(x_1, \dots, x_n) \rightarrow \\ &\rightarrow H^iK(x_1, \dots, x_{n-1}) \xrightarrow{-x_n} H^iK(x_1, \dots, x_{n-1}) \rightarrow \dots \end{aligned} \quad (20.2)$$

Proof. Let e_1, \dots, e_n be the basis of R^n . Then $R_{e_1, \dots, e_n}^n = R_{e_1, \dots, e_{n-1}}^{n-1} \oplus R_{e_n}$, and

$$K^k(x_1, \dots, x_n) = \Lambda^k R^n \simeq \Lambda^k R^{n-1} \oplus \Lambda^{k-1} R^{n-1} = K^{k-1}(x_1, \dots, x_{n-1}) \oplus K^k(x_1, \dots, x_{n-1})$$

More precisely, $e_{i_1} \wedge \dots \wedge e_{i_k} \in \Lambda^k R^{n-1}$ is mapped to $e_{i_1} \wedge \dots \wedge e_{i_k} \in \Lambda^k R^n$, while $e_{i_1} \wedge \dots \wedge e_{i_{k-1}} \in \Lambda^{k-1} R^{n-1}$ goes to $e_n \wedge e_{i_1} \wedge \dots \wedge e_{i_{k-1}} \in \Lambda^k R^n$. Under this identification the Koszul differential in $K(x_1, \dots, x_n)$, i.e. exterior multiplication with $x_1 e_1 + \dots + x_n e_n$, has a block matrix

$$\begin{bmatrix} d & 0 \\ -\alpha & d \end{bmatrix},$$

where d is the Koszul differential in $K(x_1, \dots, x_{n-1})$, i.e. exterior multiplication with $x_1 e_1 + \dots + x_{n-1} e_{n-1}$. This agrees with the differential (19.2) of the mapping cone (shifted by -1). The long exact sequence follows from Lemma 19.3. \square

This exact sequence is all we need to understand cohomology of the Koszul complex.

20.3. LEMMA. If x_1, \dots, x_n is a regular sequence then $H^i K(x_1, \dots, x_n) = 0$ for $i < n$.

Proof. By induction, we have $H^i K(x_1, \dots, x_{n-1}) = 0$ for $i < n-1$. By (20.2), this implies that $H^i K(x_1, \dots, x_n) = 0$ for $i < n-1$. Also, we see that $H^{n-1} K(x_1, \dots, x_n)$ is isomorphic to the kernel of the map

$$H^{n-1} K(x_1, \dots, x_{n-1}) \xrightarrow{-x_n} H^{n-1} K(x_1, \dots, x_{n-1}),$$

but this can be identified with

$$R/(x_1, \dots, x_{n-1}) \xrightarrow{-x_n} R/(x_1, \dots, x_{n-1}).$$

Since x_n is not a zero-divisor modulo (x_1, \dots, x_{n-1}) , the latter map is injective. \square

20.4. LEMMA. Assume R is local Noetherian and x_1, \dots, x_n belong to the maximal ideal. If $H^k K(x_1, \dots, x_n) = 0$ then $H^j K(x_1, \dots, x_n) = 0$ for $j < k$.

Proof. By (20.2), we have a surjection

$$H^{k-1}K(x_1, \dots, x_{n-1}) \xrightarrow{-x_n} H^{k-1}K(x_1, \dots, x_n),$$

i.e. $H^{k-1}K(x_1, \dots, x_n) = x_n H^{k-1}K(x_1, \dots, x_{n-1})$. By Nakayama's lemma, this implies $H^{k-1}K(x_1, \dots, x_{n-1}) = 0$. Induction on n gives $H^j K(x_1, \dots, x_{n-1}) = 0$ for any $j < k$. A final application of (20.2) gives $H^j K(x_1, \dots, x_n) = 0$ for $j < k$. \square

Proof of Theorem 18.15. It remains to show that if $H^{n-1}K(x_1, \dots, x_n) = 0$ then x_1, \dots, x_n is a regular sequence (assuming R is Noetherian). As in the previous proof, we get $H^{n-2}K(x_1, \dots, x_{n-1}) = 0$. By induction this implies that x_1, \dots, x_{n-1} is a regular sequence. Finally, consider the following part of (20.2):

$$0 \rightarrow H^{n-1}K(x_1, \dots, x_{n-1}) \xrightarrow{-x_n} H^{n-1}K(x_1, \dots, x_n),$$

which can be identified with

$$0 \rightarrow R/(x_1, \dots, x_{n-1}) \xrightarrow{-x_n} R/(x_1, \dots, x_n).$$

This shows that x_n is not a zero-divisor modulo (x_1, \dots, x_{n-1}) . \square

The proof of Theorem 18.10 is finished in the exercises.

§21. DERIVED FUNCTORS. FEB 24.

Let $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ be a functor between Abelian categories. If you are uncomfortable with Abelian categories, just take $\mathcal{A} = R\text{-mod}$, $\mathcal{B} = S\text{-mod}$. We assume that \mathcal{F} is *additive*, i.e. $\text{Hom}(X, Y) \xrightarrow{\mathcal{F}} \text{Hom}(\mathcal{F}X, \mathcal{F}Y)$ is a homomorphism of Abelian groups for any objects X, Y of \mathcal{A} . We will also assume that \mathcal{F} is left- or right-exact. If \mathcal{F} is left-exact, we will construct its right derived functors $R^n \mathcal{F}$, $n \geq 0$. If \mathcal{F} is right-exact, we will construct its left derived functors $L_n \mathcal{F}$, $n \geq 0$. Construction of derived functors requires that we fix an injective (or projective) resolution for each object $A \in \mathcal{A}$. Namely, if \mathcal{F} is covariant and left-exact (or contravariant and right-exact), we will need injective resolutions. Otherwise, we will need projective resolutions. In particular, \mathcal{A} should have enough injective (or projective) objects.

21.1. LEMMA–DEFINITION. *Take an object $A \in \mathbf{Ob}(\mathcal{A})$.*

Suppose \mathcal{F} is left-exact and covariant. Take an injective resolution of A , $0 \rightarrow A \rightarrow I_0 \rightarrow I_1 \rightarrow \dots$, truncate, and apply \mathcal{F} : $\mathcal{F}(I_0) \rightarrow \mathcal{F}(I_1) \rightarrow \dots$. Then $R^n \mathcal{F}(A)$ is the n -th cohomology of this complex.

Suppose \mathcal{F} is right-exact and covariant. Take a projective resolution of A , $\dots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0$, truncate, and apply \mathcal{F} : $\dots \rightarrow \mathcal{F}(P_1) \rightarrow \mathcal{F}(P_0)$. Then $L_n \mathcal{F}(A)$ is the n -th homology of this complex.

Suppose \mathcal{F} is left-exact and contravariant. Take a projective resolution of A , $\dots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0$, truncate, and apply \mathcal{F} : $\mathcal{F}(P_0) \rightarrow \mathcal{F}(P_1) \rightarrow \dots$. Then $R^n \mathcal{F}(A)$ is the n -th cohomology of this complex.

Suppose \mathcal{F} is right-exact and contravariant. Take an injective resolution of A , $0 \rightarrow A \rightarrow I_0 \rightarrow I_1 \dots$, truncate, and apply \mathcal{F} : $\dots \rightarrow \mathcal{F}(I_1) \rightarrow \mathcal{F}(I_0)$. Then $L_n \mathcal{F}(A)$ is the n -th homology of this complex.

For any morphism $f : A \rightarrow B$, we define $R^n(f) : R^n(A) \rightarrow R^n(B)$ (resp. $L_n(f) : L_n(A) \rightarrow L_n(B)$) as follows: as in the proof of Theorem 12.6, we can find a

(co)chain map from the resolution of A to the resolution of B , and any two such maps are homotopically equivalent. Applying the functor \mathcal{F} gives a map of (co)chain complexes used to compute $R^n(A)$ and $R^n(B)$ (resp. $L_n(A)$ and $L_n(B)$). Descent to (co)homology gives a map $R^n(f) : R^n(A) \rightarrow R^n(B)$ (resp. $L_n(f) : L_n(A) \rightarrow L_n(B)$) which does not depend on any choices.

21.2. NOTATION. We will discuss some general properties of derived functors. The four cases considered above are treated in the same way, so until the rest of this section I am going to assume that

\mathcal{F} is covariant and right exact.

unless specified otherwise.

Note that our definition of $L_n\mathcal{F}(A)$ depends on the choice of a projective resolution P_\bullet of A , which we fixed. In practice, we may want to use another resolution P'_\bullet . As in the Lemma-Definition, P_\bullet and P'_\bullet are homotopy equivalent. Indeed, we have chain maps $P_\bullet \rightarrow P'_\bullet$ and $P'_\bullet \rightarrow P_\bullet$ and any two such maps are homotopy equivalent. In particular, compositions $P_\bullet \rightarrow P'_\bullet \rightarrow P_\bullet$ and $P'_\bullet \rightarrow P_\bullet \rightarrow P'_\bullet$ are homotopy equivalent to identity maps. Applying \mathcal{F} to all maps above gives canonical (i.e. independent on any choices above) homotopy equivalence of complexes $\mathcal{F}(P_\bullet)$ and $\mathcal{F}(P'_\bullet)$. In particular, there exists a canonical isomorphism

$$H_n(\mathcal{F}(P_\bullet)) \simeq H_n(\mathcal{F}(P'_\bullet)). \quad (21.3)$$

In practice, this implies that very little depends on the choice of a projective resolution.

21.4. LEMMA. $L_0\mathcal{F} \simeq \mathcal{F}$. (If \mathcal{F} is left-exact then $R^0\mathcal{F} \simeq \mathcal{F}$.)

Proof. By definition, $L_0\mathcal{F}(A)$ is the cokernel of the map $\mathcal{F}(P_1) \rightarrow \mathcal{F}(P_0)$. But we have an exact sequence $P_1 \rightarrow P_0 \rightarrow A \rightarrow 0$, so by right-exactness of \mathcal{F} , we have $L_0\mathcal{F}(A) \simeq \mathcal{F}(A)$. A morphism $A \rightarrow B$ induces a commutative diagram

$$\begin{array}{ccccccc} P_1 & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ Q_1 & \longrightarrow & Q_0 & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

(unique up to homotopy). Applying \mathcal{F} to it shows (after a little diagram chasing) that we have a commutative square

$$\begin{array}{ccc} L_0\mathcal{F}(A) & \longrightarrow & \mathcal{F}(A) \\ \downarrow & & \downarrow \\ L_0\mathcal{F}(B) & \longrightarrow & \mathcal{F}(B) \end{array}$$

So in fact we have a natural isomorphism from $L_0\mathcal{F}$ to \mathcal{F} . □

This is simple but often useful:

21.5. LEMMA. If A is projective then $L_n\mathcal{F}(A) = 0$ for $n > 0$.

Proof. A projective resolution is $\dots \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow A \rightarrow A \rightarrow 0$. □

§22. δ -FUNCTORS. FEB 27.

The most important fact about derived functors is how they operate on short exact sequences. An abstract framework was introduced by Grothendieck:

22.1. DEFINITION. Let \mathcal{A}, \mathcal{B} be abelian categories. A *homological* (covariant) δ -functor is a sequence of additive functors $T_n : \mathcal{A} \rightarrow \mathcal{B}$ along with a morphism $\delta : T_n(A'') \rightarrow T_{n-1}(A')$ for each short exact sequence

$$0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$$

such that the following two axioms are satisfied.

- For each short exact sequence as above, we have an induced long exact sequence

$$\dots \rightarrow T_n(A') \rightarrow T_n(A) \rightarrow T_n(A'') \xrightarrow{\delta} T_{n-1}(A') \rightarrow T_{n-1}(A) \rightarrow \dots$$

- This sequence is natural in the following sense: for each commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \end{array} \quad (22.2)$$

with exact rows, the induced diagram

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & T_n(A') & \longrightarrow & T_n(A) & \longrightarrow & T_n(A'') & \xrightarrow{\delta} & T_{n-1}(A') & \longrightarrow & T_{n-1}(A) & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & T_n(B') & \longrightarrow & T_n(B) & \longrightarrow & T_n(B'') & \xrightarrow{\delta} & T_{n-1}(B') & \longrightarrow & T_{n-1}(B) & \longrightarrow & \dots \end{array}$$

is also commutative. Of course only commutativity of squares with δ 's does not immediately follow from functoriality.

A *cohomological* δ -functor is defined similarly (δ will be a morphism $T^n(A'') \rightarrow T^{n+1}(A')$). One can also define contravariant δ -functors in an obvious way.

The main example is homology of a chain complex. More precisely, let $\mathbf{Kom}_{\geq 0}(\mathcal{A})$ be the category of chain complexes of elements of an abelian category \mathcal{A} concentrated in non-negative degrees. We have homology functors

$$H_n : \mathbf{Kom}_{\geq 0}(\mathcal{A}) \rightarrow \mathcal{A}, \quad n \geq 0.$$

22.3. LEMMA. H_n is a homological δ -functor.

Proof. We have to show that the connecting homomorphism δ is natural on short exact sequences. So consider a commutative diagram of maps of complexes with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A'_\bullet & \xrightarrow{\alpha'} & A_\bullet & \xrightarrow{\alpha''} & A''_\bullet & \longrightarrow & 0 \\ & & f' \downarrow & & \downarrow f & & \downarrow f'' & & \\ 0 & \longrightarrow & B'_\bullet & \xrightarrow{\beta'} & B_\bullet & \xrightarrow{\beta''} & B''_\bullet & \longrightarrow & 0 \end{array}$$

We have to show that the diagram

$$\begin{array}{ccc}
 H_n(A'') & \xrightarrow{\delta} & H_{n-1}(A') \\
 f'' \downarrow & & \downarrow f' \\
 H_n(B'') & \xrightarrow{\delta} & H_{n-1}(B')
 \end{array} \tag{22.4}$$

Take a class $[a'']$ in $H_n(A'')$ represented by $a'' \in A''_n$ with $\partial a'' = 0$. Then $a'' = \alpha''(a)$ for some $a \in A_n$. We can find $a' \in A_{n-1}$ such that $\alpha'(a') = a$. Recall that $\delta([a'']) = [a']$. And so $f'(\delta([a''])) = [f'(a')]$. On the other hand, let $b'' = f''(a'')$, $b = f(a)$, and $b' = f'(a')$. Then $\partial b'' = 0$, $b'' = \beta''(b)$, $\beta'(b') = b$ by commutativity of the diagram of complexes. Therefore $\delta[b''] = [b']$. So the diagram (22.4) is commutative. \square

Now the main result:

22.5. THEOREM. *Let \mathcal{F} be a covariant right-exact functor (other cases are similar). Then left derived functors $L_n\mathcal{F}$ form a homological δ -functor.*

Proof. For each short exact sequence

$$0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$$

we have to construct a connecting homomorphism $L_n\mathcal{F}(A'') \xrightarrow{\delta} L_{n-1}\mathcal{F}(A')$, show that they fit into a long exact sequence

$$\dots \rightarrow L_n\mathcal{F}(A') \rightarrow L_n\mathcal{F}(A) \rightarrow L_n\mathcal{F}(A'') \xrightarrow{\delta} L_{n-1}\mathcal{F}(A') \rightarrow L_{n-1}\mathcal{F}(A) \rightarrow \dots, \tag{22.6}$$

and check naturality of δ on commutative diagrams (22.2).

Chosen projective resolutions for A' and A'' fit into a “horse-shoe diagram”

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \dots & \longrightarrow & P'_1 & \longrightarrow & P'_0 & \longrightarrow & A' \longrightarrow 0 \\
 & & & & & & \downarrow \\
 & & & & & & A \longrightarrow 0 \\
 & & & & & & \downarrow \\
 \dots & \longrightarrow & P''_1 & \longrightarrow & P''_0 & \longrightarrow & A'' \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

22.7. CLAIM. We can extend this (not uniquely) to a commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \dots & \longrightarrow & P'_1 & \xrightarrow{\partial'_1} & P'_0 & \xrightarrow{\partial'_0} & A' \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \dots & \longrightarrow & P'_1 \oplus P''_1 & \xrightarrow{\partial_1} & P'_0 \oplus P''_0 & \xrightarrow{\partial_0} & A \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \dots & \longrightarrow & P''_1 & \xrightarrow{\partial''_0} & P''_0 & \xrightarrow{\partial''_0} & A'' \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

where the vertical arrows are obvious inclusions/projections and the middle row is a projective resolution.

Proof of the Claim. Commutativity of squares on the right implies that $\partial_0|_{P'_0} = \partial'_0$ and $\partial_0|_{P''_0}$ should be equal to the lift of the map $P''_0 \xrightarrow{\partial''_0} A''$ to A , which exists by projectivity of P''_0 . A diagram chase shows that ∂_0 is surjective. Commutativity of the remaining squares can be expressed by writing ∂_n , $n \geq 1$, as a block matrix

$$\begin{bmatrix} \partial'_n & \lambda_n \\ 0 & \partial''_n \end{bmatrix}$$

It remains to choose λ_n inductively such that the middle row of the horse-shoe diagram is exact. This is left as a homework exercise. \square

Next we truncate the horse-shoe diagram and apply \mathcal{F} to it:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 \dots & \longrightarrow & \mathcal{F}(P'_1) & \longrightarrow & \mathcal{F}(P'_0) & & \\
 & & \downarrow & & \downarrow & & \\
 \dots & \longrightarrow & \mathcal{F}(P'_1) \oplus \mathcal{F}(P''_1) & \longrightarrow & \mathcal{F}(P'_0) \oplus \mathcal{F}(P''_0) & & \\
 & & \downarrow & & \downarrow & & \\
 \dots & \longrightarrow & \mathcal{F}(P''_1) & \longrightarrow & \mathcal{F}(P''_0) & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

Here we use a homework exercise that additive functors commute with direct sums. So columns stay exact, i.e. we have a short exact sequence of complexes

$$0 \rightarrow \mathcal{F}(P'_\bullet) \rightarrow \mathcal{F}(P_\bullet) \rightarrow \mathcal{F}(P''_\bullet) \rightarrow 0,$$

where P_\bullet is the middle projective resolution that we have constructed. This induces a long exact sequence in homology

$$\dots \rightarrow L_n \mathcal{F}(A') \rightarrow H_n(\mathcal{F}(P_\bullet)) \rightarrow L_n \mathcal{F}(A'') \xrightarrow{\delta} L_{n-1} \mathcal{F}(A') \rightarrow H_{n-1}(\mathcal{F}(P_\bullet)) \rightarrow \dots,$$

where we write $H_n(\mathcal{F}(P_\bullet))$ instead of $L_n \mathcal{F}(A)$ because P_\bullet could be different from a “chosen” projective resolution of A . However, natural isomorphisms (21.3) allow us to construct (22.6). Notice, however, that at this point it is not clear that δ is canonical and does not depend on the construction of P_\bullet . This will follow from our proof of naturality. So consider a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & f' \downarrow & & f \downarrow & & f'' \downarrow & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \end{array}$$

Applying our “horseshoe” construction and lifting maps f, f' to maps of projective resolutions gives a commutative diagram of projective resolutions.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P'_\bullet & \longrightarrow & P_\bullet & \longrightarrow & P''_\bullet & \longrightarrow & 0 \\ & & F' \downarrow & & & & F'' \downarrow & & \\ 0 & \longrightarrow & Q'_\bullet & \longrightarrow & Q_\bullet & \longrightarrow & Q''_\bullet & \longrightarrow & 0 \end{array} \quad (22.8)$$

22.9. CLAIM. *There exists a chain map $F : P_\bullet \rightarrow Q_\bullet$ lifting f that makes (22.8) into a commutative diagram.*

Given the claim, let’s finish the proof of the theorem. As above, applying the functor \mathcal{F} gives a commutative diagram of chain complexes with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}(P'_\bullet) & \longrightarrow & \mathcal{F}(P_\bullet) & \longrightarrow & \mathcal{F}(P''_\bullet) & \longrightarrow & 0 \\ & & \mathcal{F}(F') \downarrow & & \mathcal{F}(F) \downarrow & & \mathcal{F}(F'') \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}(Q'_\bullet) & \longrightarrow & \mathcal{F}(Q_\bullet) & \longrightarrow & \mathcal{F}(Q''_\bullet) & \longrightarrow & 0 \end{array}$$

Now applying Lemma 22.3 gives a required commutative diagram

$$\begin{array}{ccccccccccccccc} \dots & \longrightarrow & L_n \mathcal{F}(A') & \longrightarrow & L_n \mathcal{F}(A) & \longrightarrow & L_n \mathcal{F}(A'') & \xrightarrow{\delta} & L_{n-1} \mathcal{F}(A') & \longrightarrow & L_{n-1} \mathcal{F}(A) & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & L_n \mathcal{F}(B') & \longrightarrow & L_n \mathcal{F}(B) & \longrightarrow & L_n \mathcal{F}(B'') & \xrightarrow{\delta} & L_{n-1} \mathcal{F}(B') & \longrightarrow & L_{n-1} \mathcal{F}(B) & \longrightarrow & \dots \end{array}$$

To show that δ is independent on various choices, we use a trick: suppose we have two different short exact sequences of projective resolutions

$$0 \rightarrow P'_\bullet \rightarrow P_\bullet \rightarrow P''_\bullet \rightarrow 0 \quad \text{and} \quad 0 \rightarrow P'_\bullet \rightarrow \tilde{P}_\bullet \rightarrow P''_\bullet \rightarrow 0$$

over a short exact sequence $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$. Applying the Claim to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P'_\bullet & \longrightarrow & P_\bullet & \longrightarrow & P''_\bullet & \longrightarrow & 0 \\ & & \text{Id}_{P'} \downarrow & & & & \text{Id}_{P''} \downarrow & & \\ 0 & \longrightarrow & P'_\bullet & \longrightarrow & \tilde{P}_\bullet & \longrightarrow & P''_\bullet & \longrightarrow & 0 \end{array}$$

and applying Lemma 22.3 gives a commutative diagram

$$\begin{array}{ccccccccc}
 \dots & \longrightarrow & L_n \mathcal{F}(A') & \longrightarrow & L_n \mathcal{F}(A) & \longrightarrow & L_n \mathcal{F}(A'') & \xrightarrow{\delta} & L_{n-1} \mathcal{F}(A') & \longrightarrow & L_{n-1} \mathcal{F}(A) & \longrightarrow & \dots \\
 & & = \downarrow & & = \downarrow & & = \downarrow & & = \downarrow & & = \downarrow & & \\
 \dots & \longrightarrow & L_n \mathcal{F}(A') & \longrightarrow & L_n \mathcal{F}(A) & \longrightarrow & L_n \mathcal{F}(A'') & \xrightarrow{\tilde{\delta}} & L_{n-1} \mathcal{F}(A') & \longrightarrow & L_{n-1} \mathcal{F}(A) & \longrightarrow & \dots
 \end{array}$$

So $\delta = \tilde{\delta}$. It remains to prove the claim. □

Proof of the Claim. To keep (22.8) commutative, F should be given by a block matrix

$$F = \begin{bmatrix} F'_n & \gamma_n \\ 0 & F''_n \end{bmatrix}$$

Recall that differentials in P and Q are also given by block matrices

$$\partial_n^P = \begin{bmatrix} \partial'_n & \lambda_n \\ 0 & \partial''_n \end{bmatrix} \quad \text{and} \quad \partial_n^Q = \begin{bmatrix} \partial'_n & \mu_n \\ 0 & \partial''_n \end{bmatrix}$$

(we do not distinguish notationally differentials in P' and Q' (resp. P'' and Q'')). We leave it as a homework exercise to construct γ_0 so that F lifts f . The remaining condition is that

$$\partial^Q F = F \partial^P.$$

This can be written as follows:

$$\partial'_n \gamma_n + \mu_n F''_n = F'_{n-1} \lambda_n + \gamma_{n-1} \partial''_n.$$

We are going to show this by induction. The base case $n = 0$ is left as an exercise. We do the inductive step, i.e. construct γ_{n+1} so that

$$\partial'_{n+1} \gamma_{n+1} = g_{n+1} := -\mu_{n+1} F''_{n+1} + F'_n \lambda_{n+1} + \gamma_n \partial''_{n+1}.$$

By projectivity, of P''_{n+1} , to construct γ_{n+1} it is enough to show that

$$\text{Im}(g_{n+1}) \subset \text{Im}(\partial'_{n+1}) = \text{Ker}(\partial_n).$$

So it suffices to prove that $\partial'_n g_{n+1} = 0$. We compute

$$\begin{aligned}
 & -\partial'_n \mu_{n+1} F''_{n+1} + \partial'_n F'_n \lambda_{n+1} + \partial'_n \gamma_n \partial''_{n+1} = \\
 & -\partial'_n \mu_{n+1} F''_{n+1} + \partial'_n F'_n \lambda_{n+1} + (-\mu_n F''_n + F'_{n-1} \lambda_n + \gamma_{n-1} \partial''_n) \partial''_{n+1} = \\
 & -\partial'_n \mu_{n+1} F''_{n+1} + \partial'_n F'_n \lambda_{n+1} - \mu_n F''_n \partial''_{n+1} + F'_{n-1} \lambda_n \partial''_{n+1} = \\
 & -\partial'_n \mu_{n+1} F''_{n+1} + F'_{n-1} \partial'_n \lambda_{n+1} - \mu_n \partial''_{n+1} F''_{n+1} + F'_{n-1} \lambda_n \partial''_{n+1} = \\
 & -(\partial'_n \mu_{n+1} + \mu_n \partial''_{n+1}) F''_{n+1} + F'_{n-1} (\partial'_n \lambda_{n+1} + \lambda_n \partial''_{n+1}) = 0
 \end{aligned}$$

because

$$\begin{bmatrix} \partial'_n & \mu_n \\ 0 & \partial''_n \end{bmatrix} \begin{bmatrix} \partial'_{n+1} & \mu_{n+1} \\ 0 & \partial''_{n+1} \end{bmatrix} = \begin{bmatrix} \partial'_n & \lambda_n \\ 0 & \partial''_n \end{bmatrix} \begin{bmatrix} \partial'_{n+1} & \lambda_{n+1} \\ 0 & \partial''_{n+1} \end{bmatrix} = 0$$

QED □

§23. Tor. FEB 29.

We fix an R -module B and consider a right exact functor $R\text{-mod} \xrightarrow{\bullet \otimes_R B} R\text{-mod}$. Its left-derived functors are called Tor-functors:

$$\text{Tor}_n^R(\bullet, B) = L_n(\bullet \otimes_R B).$$

Concretely, using a fixed projective resolution

$$\dots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0,$$

$\text{Tor}_n^R(A, B)$ is the n -th homology group of the complex

$$\dots \rightarrow P_1 \otimes_R B \rightarrow P_0 \otimes_R B.$$

23.1. EXAMPLE. Let $R = k[x, y]$, $A = R/(x)$, $B = R/(y)$. Using Koszul resolution

$$\dots \rightarrow 0 \rightarrow R \xrightarrow{x} R \rightarrow A \rightarrow 0,$$

$\text{Tor}_n^R(A, B)$ is the n -th homology group of the complex

$$\dots \rightarrow 0 \rightarrow R \otimes_R B \xrightarrow{x \otimes \text{Id}} R \otimes_R B,$$

i.e.

$$\dots \rightarrow 0 \rightarrow B \xrightarrow{x} B.$$

So $\text{Tor}_0(A, B) = R/(x, y) \simeq k$ and $\text{Tor}_1(A, B) = 0$.

As with any right-exact functor, $\text{Tor}_0^R(A, B) \simeq A \otimes_R B$. Applying Theorem 22.5 gives a long exact sequence

$$\dots \rightarrow \text{Tor}_n^R(A, B) \rightarrow \text{Tor}_n^R(A', B) \xrightarrow{\delta} \text{Tor}_{n-1}^R(A', B) \rightarrow \text{Tor}_{n-1}^R(A, B) \rightarrow \dots$$

for each short exact sequence

$$0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$$

We can also study dependence of Tor on the second argument:

23.2. LEMMA. An exact sequence $0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0$ induces an exact sequence

$$\dots \rightarrow \text{Tor}_i(A, B') \rightarrow \text{Tor}_i(A, B) \rightarrow \text{Tor}_i(A, B'') \rightarrow \text{Tor}_{i-1}(A, B') \rightarrow \dots \quad (23.3)$$

Proof. A projective resolution

$$\dots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0$$

induces a commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \dots & \longrightarrow & P_2 \otimes B' & \longrightarrow & P_1 \otimes B' & \longrightarrow & P_0 \otimes B' \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \dots & \longrightarrow & P_2 \otimes B & \longrightarrow & P_1 \otimes B & \longrightarrow & P_0 \otimes B \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \dots & \longrightarrow & P_2 \otimes B'' & \longrightarrow & P_1 \otimes B'' & \longrightarrow & P_0 \otimes B'' \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Since projective modules are flat (a homework exercise), columns of this diagram are exact, i.e. we have a short exact diagram of chain complexes

$$0 \rightarrow P_\bullet \otimes B' \rightarrow P_\bullet \otimes B \rightarrow P_\bullet \otimes B'' \rightarrow 0$$

The long exact sequence in homology gives (23.3). □

§24. Ext.

We fix an R -module B and consider a left exact contravariant functor $R\text{-mod} \xrightarrow{\text{Hom}_R(\bullet, B)} R\text{-mod}$. Its right-derived functors are called Ext-functors:

$$\text{Ext}_R^n(\bullet, B) = R^n(\text{Hom}_R(\bullet, B)).$$

Concretely, using a fixed projective resolution

$$\dots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0,$$

$\text{Ext}_R^n(A, B)$ is the n -th cohomology group of the complex

$$\text{Hom}_R(P_0, B) \rightarrow \text{Hom}_R(P_1, B) \rightarrow \dots$$

As with any left-exact functor, $\text{Ext}_R^0(A, B) \simeq \text{Hom}_R(A, B)$.

Since $\text{Ext}_R^n(\bullet, B)$ is a cohomological contravariant δ -functor, Theorem 22.5 gives a long exact sequence

$$\dots \rightarrow \text{Ext}_R^n(A, B) \rightarrow \text{Ext}_R^n(A', B) \xrightarrow{\delta} \text{Ext}_R^{n+1}(A'', B) \rightarrow \text{Ext}_R^{n+1}(A, B) \rightarrow \dots$$

for each short exact sequence

$$0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$$

We can also study dependence of Ext on the second argument:

24.1. LEMMA. An exact sequence $0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0$ induces an exact sequence

$$\dots \rightarrow \text{Ext}^i(A, B') \rightarrow \text{Ext}^i(A, B) \rightarrow \text{Ext}^i(A, B'') \rightarrow \text{Ext}^{i+1}(A, B') \rightarrow \dots$$

Proof. Analogous to Lemma 23.2. □

24.2. REMARK. We defined Tor and Ext only in the category $R\text{-mod}$, which makes sense for Tor because the tensor product is not part of the set-up of an abelian category. However, $\text{Hom}(\bullet, B)$ is a well-defined left-exact functor $\mathcal{A} \rightarrow \mathbf{Ab}$ for any Abelian category \mathcal{A} . In particular, we can define $\text{Ext}^n(A, B)$ in any abelian category, as soon as we have enough projectives.

§25. Ext^1 AND EXTENSIONS. MAR 2.

25.1. THEOREM. Fix R -modules A and B (or more generally objects of an abelian category with enough projectives). There is a bijection (described in the proof) between elements of $\text{Ext}^1(A, B)$ and isomorphism classes of extensions, i.e. short exact sequences

$$0 \rightarrow B \rightarrow X \rightarrow A \rightarrow 0, \quad (25.2)$$

where two extensions X, X' are viewed as isomorphic if there exists an isomorphism $X \rightarrow X'$ which restricts to identity on B and induces an identity on A . Under this bijection $0 \in \text{Ext}^1(A, B)$ corresponds to the split extension

$$0 \rightarrow B \rightarrow B \oplus A \rightarrow A \rightarrow 0,$$

For example, $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}_2, \mathbb{Z}_3) = 0$ because there exists only the split extension. On the other hand, $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$, because there exists a unique non-split extension $0 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_4 \rightarrow \mathbb{Z}_2 \rightarrow 0$.

Proof. The proof is quite constructive. Given an extension (25.2), consider the following segment of the long exact sequence of Ext:

$$\text{Hom}(X, B) \rightarrow \text{Hom}(B, B) \xrightarrow{\delta} \text{Ext}^1(A, B)$$

We let $e = \delta(\text{Id}_B)$ to be the class of the extension. Notice that $e = 0$ if and only if the injection $B \hookrightarrow X$ has a section $X \rightarrow B$, i.e. when (25.2).

In an opposite direction, we start with $e \in \text{Ext}^1(A, B)$. Fix an exact sequence

$$0 \rightarrow M \xrightarrow{j} P \rightarrow A \rightarrow 0$$

with P projective. Consider the following segment of the long exact sequence of Ext:

$$\text{Hom}(P, B) \rightarrow \text{Hom}(M, B) \xrightarrow{\delta} \text{Ext}^1(A, B) \rightarrow \text{Ext}^1(P, B) = 0$$

because P is projective. This implies that there exists $\beta \in \text{Hom}(M, B)$ such that $\delta(\beta) = e$. Using β , we construct an extension using the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{j} & P & \longrightarrow & A \longrightarrow 0 \\ & & \beta \downarrow & & \downarrow & = \downarrow & \\ 0 & \longrightarrow & B & \longrightarrow & X & \longrightarrow & A \longrightarrow 0 \end{array} \quad (25.3)$$

called *the push-out*. Here

$$X = B \oplus P / (\beta(m), -j(m))_{m \in M}$$

Maps $B \rightarrow X$ and $P \rightarrow X$ are induced by inclusions into the direct sum. The map $X \rightarrow A$ is induced by the map $P \rightarrow A$. We checked that everything is well-defined and commutative. Noticed that β is not defined canonically, but if β and β' both lift $e \in \text{Ext}^1(A, B)$ then $\beta' - \beta = \rho \circ j$ for

some $\rho \in \text{Hom}(P, B)$. Extensions X and X' are going to be isomorphic. Indeed an isomorphism $B \oplus P \rightarrow B \oplus P$ with matrix

$$\begin{bmatrix} \text{Id}_B & -\rho \\ 0 & \text{Id}_P \end{bmatrix}$$

takes a subgroup generated by vectors

$$\begin{bmatrix} \beta(m) \\ -j(m) \end{bmatrix}$$

to a subgroup generated by vectors

$$\begin{bmatrix} \beta'(m) \\ -j(m) \end{bmatrix}$$

So we have well-defined maps from the set of extensions to Ext^1 and vice versa.

Start with $e \in \text{Ext}^1(A, B)$, produce an extension. Applying naturality of σ to the diagram (25.3) gives a commutative diagram

$$\begin{array}{ccc} \text{Hom}(B, B) & \xrightarrow{\delta} & \text{Ext}^1(A, B) \\ \beta \downarrow & & = \downarrow \\ \text{Hom}(M, B) & \xrightarrow{\delta} & \text{Ext}^1(A, B) \end{array}$$

So $\delta(\text{Id}_B) = \delta(\beta) = e$.

Now start with an extension (25.2) and take $e = \delta(\text{Id}(B))$. By projectivity of P , there exists a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{j} & P & \longrightarrow & A \longrightarrow 0 \\ & & \beta \downarrow & & \tau \downarrow & & = \downarrow \\ 0 & \longrightarrow & B & \xrightarrow{i} & X & \longrightarrow & A \longrightarrow 0 \end{array}$$

We only have to show that in this case X is isomorphic to a push-out. Indeed, τ and i give a map $B \oplus P \rightarrow X$. Chasing the diagram shows that this map is surjective and that its kernel is given by vectors $\begin{bmatrix} \beta(m) \\ -j(m) \end{bmatrix}$. \square

§26. SPECTRAL SEQUENCE. MAR 5

26.1. DEFINITION. A *spectral sequence* $\{E_r, d_r\}$, $r \geq 0$, is a sequence of bi-graded objects

$$E_r = \bigoplus_{p,q \geq 0} E_r^{p,q}$$

of an abelian category together with *differentials*

$$d_r : E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$$

such that

- $d_r^2 = 0$.
- E_{r+1} is a homology of (E_r, d_r) , i.e. $E_{r+1} = \text{Ker } d_r / \text{Im } d_r$.

In practice we almost always have $d_r = d_{r+1} = \dots = 0$ for $r \geq r_0$, and therefore $E_r \simeq E_{r+1} \simeq \dots$ (canonical isomorphism). In this case we define $E_\infty = \bigoplus_{p,q \geq 0} E_\infty^{p,q}$ to be this common bigraded object and say that the spectral sequence *abuts* to E_∞ .

Pictures of pages of a spectral sequence E_0 has differentials upward, E_1 has differentials going to the right, differentials in E_2 follow the “knight move” (two squares to the right and one square down).

§27. SPECTRAL SEQUENCE OF A FILTERED COMPLEX. MAR 7.

Let (K^\bullet, d) be a cochain complex of elements of an abelian category such that $K^q = 0$ for $q < 0$. We fix a decreasing filtration

$$K^\bullet = F^0 K^\bullet \supset F^1 K^\bullet \supset F^2 K^\bullet \supset \dots$$

by subcomplexes. We make a *finiteness assumption*: for any n there exists p_0 such that $F^p K^n = 0$ for $p \geq p_0$. Subquotients of the filtration

$$\mathrm{Gr}^p K^\bullet = F^p K^\bullet / F^{p+1} K^\bullet$$

are also complexes (with induced differentials). Inclusions $F^p K^\bullet \hookrightarrow K^\bullet$ induce maps of cohomology

$$i_{pn} : H^n(F^p K^\bullet) \rightarrow H^n(K^\bullet)$$

This gives a filtration

$$H^n(K^\bullet) = F^0 H^n(K^\bullet) \supset F^1 H^n(K^\bullet) \supset F^2 H^n(K^\bullet) \supset \dots,$$

where

$$F^p H^n = \mathrm{Im} i_{pn}.$$

We denote the subquotients by

$$\mathrm{Gr}^p H^n = F^p H^n / F^{p+1} H^n.$$

27.1. THEOREM. *There exists a spectral sequence $\{E_r, d_r\}$ with*

$$E_0^{p,q} = \mathrm{Gr}^p K^{p+q}, \quad d_0 \text{ induced by } d;$$

$$E_1^{p,q} = H^{p+q}(\mathrm{Gr}^p K^\bullet),$$

d_1 is the connecting homomorphism in the long exact sequence associated with a short exact sequence of complexes

$$0 \rightarrow F^{p+1} K^{p+\bullet} / F^{p+2} K^{p+\bullet} \rightarrow F^p K^{p+\bullet} / F^{p+2} K^{p+\bullet} \rightarrow F^p K^{p+\bullet} / F^{p+1} K^{p+\bullet} \rightarrow 0$$

This spectral sequence “abuts” to $H(K^\bullet)$ in the following sense: for any p and q , we have an eventual stabilization $E_r^{pq} = E_{r+1}^{pq} = \dots =: E_\infty^{pq}$ and

$$E_\infty^{pq} = \mathrm{Gr}^p(H^{p+q}(K)).$$

Proof. We followed closely exposition in Voisin’s *Hodge Theory and Complex Algebraic Geometry*, page 201. \square

27.2. EXAMPLE. A two-term filtration $K = F^0 K \supset F^1 K = K'$ is equivalent to a short exact sequence of complexes

$$0 \rightarrow K' \rightarrow K \rightarrow K'' \rightarrow 0.$$

We checked in class that in this case the spectral sequence is equivalent to the long exact sequence in cohomology.

§28. SPECTRAL SEQUENCE OF A DOUBLE COMPLEX. MAR 9.

Consider a double complex $K^{p,q}$, $p, q \geq 0$ with a horizontal differential d' and a vertical differential d'' . We denote by K^n the total complex $K^n = \bigoplus_{p+q=n} K^{p,q}$ with differential $d = d' + d''$. It has a “horizontal filtration” defined by

$$F^p K^n = \bigoplus_{\substack{p'+q=n \\ p' \geq p}} K^{p',q}.$$

Let’s work out the spectral sequence. Since $F^p K^{p+q} / F^{p+1} K^{p+q} = K^{p,q}$ and $d'(F^p K) \subset F^{p+1} K$, we have

$$E_0^{p,q} = K^{p,q}, \quad d_0 = d''$$

and

$$E_1^{p,q} = H_{d''}^{p,q}.$$

To figure out d_1 , consider a short exact sequence

$$0 \rightarrow F^{p+1} K^{p+\bullet} / F^{p+2} K^{p+\bullet} \rightarrow F^p K^{p+\bullet} / F^{p+2} K^{p+\bullet} \rightarrow F^p K^{p+\bullet} / F^{p+1} K^{p+\bullet} \rightarrow 0$$

The complex in the middle is the sum of two columns of the double complex, the left complex is the inclusion of the left column, and the right complex is a projection onto the left column. To compute d_1 , we compute the connecting homomorphism by chasing the diagram: take $[\alpha] \in H_{d''}^{p,q}$. α as an element of $K^{p,q}$, and we can lift it to the element of the complex in the middle by adding 0 in degree $(p+1, q-1)$. Then $d(\alpha) = d'(\alpha) \in K^{p+1,q}$. So $\delta([\alpha]) = [d'(\alpha)]$, i.e. d_1 is induced by d' :

$$E_1^{p,q} = (H_{d''}^{p,q}, d').$$

By Theorem 27.1, the spectral sequence abuts to the cohomology of the total complex in the following sense: $H^n(K^\bullet)$ has a filtration $H^n = F^0 H^n \supset F^1 H^n \supset \dots \supset H^n F^n$ such that subsequent quotients are located on the n -th anti-diagonal of $E_\infty^{p,q}$ starting with a square on the top.

Notice that the double complex has another, “vertical” filtration with

$$F^q K^n = \bigoplus_{\substack{p+q'=n \\ q' \geq q}} K^{p,q'}.$$

In the corresponding spectral sequence d_0 is the horizontal differential d' and d_1 is induced by the vertical differential d'' (in particular notice that to make these differentials conform to the standard recipe of the spectral sequence, one has to reflect the first quadrant with respect to the diagonal. This spectral sequence also abuts to $H^n(K)$, although the filtration on $H^n(K)$ will be different.

§29. Ext AND Tor DEFINED USING THE SECOND ARGUMENT

Recall that $\text{Ext}^n(M, N)$ was defined as a right derived functor of a left-exact contravariant functor $\text{Hom}(\bullet, N)$. We can also consider a right derived functor $\hat{\text{Ext}}^n(M, \bullet)$ of a left-exact covariant functor $\text{Hom}(M, \bullet)$.

29.1. LEMMA. $\text{Ext}^n(M, N)$ is isomorphic to $\hat{\text{Ext}}^n(M, N)$

Proof. To define $\text{Ext}^n(M, N)$ we need a projective resolution

$$\dots \rightarrow P_1 \rightarrow P_0 \rightarrow M,$$

and to define $\hat{\text{Ext}}^n(M, N)$ we need an injective resolution

$$N \rightarrow I_0 \rightarrow I_1 \rightarrow \dots$$

We can relate them by a commutative diagram

$$\begin{array}{ccccccc}
 & \dots & & \dots & & \dots & \\
 & \uparrow & & \uparrow & & \uparrow & \\
 \text{Hom}(M, I_1) & \longrightarrow & \text{Hom}(P_0, I_1) & \longrightarrow & \text{Hom}(P_1, I_1) & \longrightarrow & \dots \\
 & \uparrow & & \uparrow & & \uparrow & \\
 \text{Hom}(M, I_0) & \longrightarrow & \text{Hom}(P_0, I_0) & \longrightarrow & \text{Hom}(P_1, I_0) & \longrightarrow & \dots \\
 & \uparrow & & \uparrow & & \uparrow & \\
 \text{Hom}(M, N) & \longrightarrow & \text{Hom}(P_0, N) & \longrightarrow & \text{Hom}(P_1, N) & \longrightarrow & \dots
 \end{array}$$

To convert it into a double complex, simply change sign of all vertical differentials in odd columns. Finally, let's remove the lower-left corner $\text{Hom}(M, N)$ from the double complex and compute cohomology of the total complex using the spectral sequence. Using the "horizontal" spectral sequence, the E_1 page is

$$\begin{array}{ccccccc}
 \hat{\text{Ext}}^2(M, N) & \longrightarrow & \dots & & \dots & & \\
 \\
 \hat{\text{Ext}}^1(M, N) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \\
 \\
 \hat{\text{Ext}}^0(M, N) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots
 \end{array}$$

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \dots$$

In the "vertical" spectral sequence, the E_1 page is

$$\begin{array}{cccc}
 \dots & & \dots & & \dots \\
 \uparrow & & \uparrow & & \uparrow \\
 0 & & 0 & & 0 & \dots \\
 \uparrow & & \uparrow & & \uparrow \\
 0 & & 0 & & 0 & \dots \\
 \uparrow & & \uparrow & & \uparrow \\
 0 & & \text{Ext}^0(M, N) & & \text{Ext}^1(M, N) & \dots
 \end{array}$$

So both spectral sequences degenerate in E_1 term and therefore $\text{Ext}^n(M, N)$ is isomorphic to $\hat{\text{Ext}}^n(M, N)$ since both are isomorphic to the $(n + 1)$ -st cohomology of the total complex. \square

Analogously we can show that

29.2. LEMMA. $\text{Tor}_n^R(M, N)$ computed as a left derived functor of $\bullet \otimes_R N$ is isomorphic to $\widehat{\text{Tor}}_n^R(M, N)$ computed as a left derived functor of $M \otimes_R \bullet$

This has an interesting corollary:

29.3. PROPOSITION. $\text{Tor}_n^R(M, N)$ is isomorphic to $\text{Tor}_n^R(N, M)$.

Proof. Choose a projective resolution $\dots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M$. Then $\text{Tor}_n^R(M, N)$ is isomorphic to the n -th homology of the complex

$$\dots \rightarrow P_2 \otimes N \rightarrow P_1 \otimes N \rightarrow P_0 \otimes N,$$

which is isomorphic to

$$\dots \rightarrow N \otimes P_2 \rightarrow N \otimes P_1 \rightarrow N \otimes P_0.$$

But the cohomology of this complex is isomorphic to $\widehat{\text{Tor}}_n^R(N, M)$, which in turn is isomorphic to $\text{Tor}_n^R(N, M)$. \square

§30. FIBER BUNDLES. MAR 12

30.1. DEFINITION. A continuous map $\pi : E \rightarrow B$ of topological spaces is called a *fiber bundle* with fiber F (or a *locally trivial fibration*) if any point of B is contained in a neighborhood U such that there exists a commutative diagram of continuous maps

$$\begin{array}{ccc} \pi^{-1}(U) & \simeq & U \times F \\ \pi \downarrow & & \downarrow pr_1 \\ U & = & U \end{array}$$

U is called a *trivializing neighborhood*.

We are going to focus on the case when F, E, B are smooth manifolds and all maps are smooth maps. We will always assume that B is connected.

In practice, one can often check that $\pi : E \rightarrow B$ is a fiber bundle by using the following theorem, which we will assume without proof.

30.2. THEOREM (Ehresmann). *Let $\pi : E \rightarrow B$ be a submersive (i.e. the differential is surjective at any point) and proper (i.e. all fibers are compact) smooth map of manifolds. Suppose B is connected. Then π is a fiber bundle.*

§31. HOMOTOPY GROUPS.

A discussion of fiber bundles will be incomplete without mentioning the long exact sequence of homotopy groups. Let $\pi : E \rightarrow B$ be a fiber bundle with fiber F . Choose a point pt in one of the fibers F . This also fixes a point in B and a point in E . Then one has the following exact sequence:

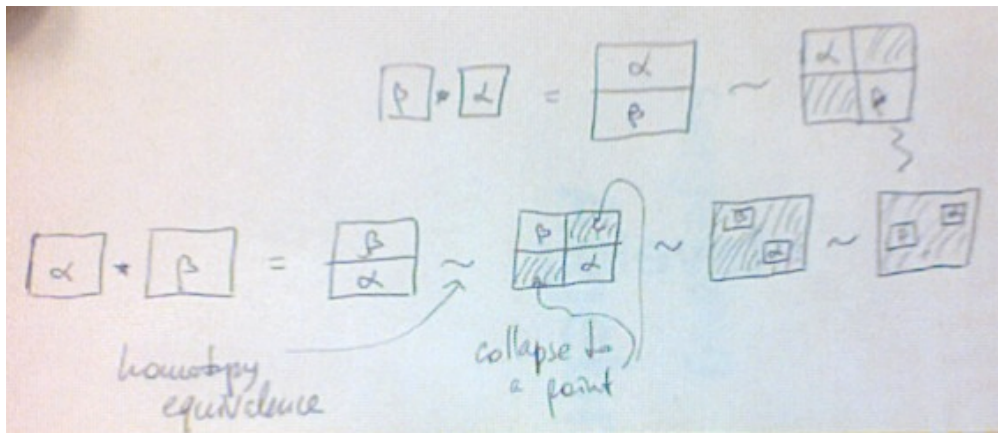
$$\dots \rightarrow \pi_2(F) \rightarrow \pi_2(E) \rightarrow \pi_2(B) \rightarrow \pi_1(F) \rightarrow \pi_1(E) \rightarrow \pi_1(B) \rightarrow \pi_0(F) \rightarrow \pi_0(E) \rightarrow \pi_0(B) \tag{31.1}$$

Here $\pi_0(X)$ is the set of connected components of X , $\pi_1(X)$ is its fundamental group, and $\pi_k(X)$, $k \geq 2$ are its higher homotopy groups. Recall

that $\pi_k(X)$ is the set of homotopy classes of maps $S^k \rightarrow X$ that send a North Pole into the fixed point $pt \in X$. This gives covariant functors

$$\pi_k : \mathbf{PointedTopologicalSpaces} \rightarrow \mathbf{Sets}.$$

In fact, $\pi_k(X)$ is a group for $k \geq 1$. To see this, notice that we can identify $\pi_k(X)$ is the set of homotopy classes of maps $[0, 1]^k \rightarrow X$ that send the boundary of the cube to the fixed point. Group operation is defined by stacking two cubes on top of each other and then shrinking the box vertically until we get a cube. The inverse in π_k is defined by flipping a cube with respect to the last coordinate. The following homotopy shows that $\pi_k(X)$ is in fact Abelian for $k \geq 2$:



As an example, let's explore *obvious homotopy groups of spheres*:

$$\begin{aligned} \pi_k(S^n) &= 0 \quad \text{if } k < n, \\ \pi_n(S^1) &= 0 \quad \text{if } n > 1, \\ \pi_n(S^n) &\simeq \mathbb{Z}. \end{aligned} \tag{31.2}$$

For the first formula, notice that any continuous map $S^k \rightarrow S^n$ is homotopy equivalent to a smooth map. If $k < n$ then this map can not be onto by Sard's theorem. So the map $S^k \rightarrow S^n$ factors through some disk $D^n \subset S^n$. But the disk is contractible, so the map is homotopy equivalent to a map to a point. In particular, S^n is simply-connected for $n > 1$.

For the second formula, notice that the universal cover of S^1 is \mathbb{R} , and since S^n is simply-connected, any map $S^n \rightarrow S^1$ lifts to a map $S^n \rightarrow \mathbb{R}$. The latter is contractible, so this map is homotopy equivalent to a map to a point.

The third formula is the least obvious. A continuous map $f : S^n \rightarrow S^n$ induces a map $\mathbb{Z} \simeq H_n(S^n, \mathbb{Z}) \xrightarrow{f_*} H_n(S^n, \mathbb{Z}) \simeq \mathbb{Z}$. As any homomorphism $\mathbb{Z} \rightarrow \mathbb{Z}$, this map should be a multiplication by $d \in \mathbb{Z}$, called *degree* of f . This gives a homomorphism $\text{deg} : \pi_n(S^n) \rightarrow \mathbb{Z}$. One can show that this map is an isomorphism.

Going back to the long exact sequence, the last three terms in (31.1) are sets, not groups. However, they are pointed sets (a connected component of the fixed point pt is distinguished). So we can define the kernel of each map as the set of elements that map to a distinguished element. So we can make

sense of exactness of (31.1) even at the last three terms. All maps in (31.1) are defined by functoriality of π_k except for connecting homomorphisms

$$\delta : \pi_k(B) \rightarrow \pi_{k-1}(F).$$

Those are defined as follows. Take an element $[f] \in \pi_k(B)$ thought of a homotopy class of a map $f : [0, 1]^k \rightarrow B$ that sends the boundary of the cube to the distinguished point. One can prove that there exists a (unique up to homotopy) map $g : [0, 1]^k \rightarrow E$ that maps one of the points in $\partial[0, 1]^k$ to the distinguished point of E and such that $\pi \circ g = f$. Notice that the boundary of the cube $\partial[0, 1]^k$ is mapped to F . In general it is no longer mapped to a point. But $\partial[0, 1]^k$ is homeomorphic to S^{k-1} , so this construction gives a well-defined class $\delta([f]) \in \pi_{k-1}(F)$.

§32. EXAMPLE: A HOPF FIBRATION

Here is a nice example that can be generalized in many ways.

A *Hopf bundle* $S^3 \rightarrow S^2$ with fiber S^1 is defined as follows. We have

$$S^3 = \{z_1, z_2 \in \mathbb{C} \mid |z_1|^2 + |z_2|^2 = 1\}.$$

$S^1 = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$ acts on S^3 by multiplying z_1, z_2 by $e^{i\theta}$. The set of orbits can be identified with a Riemann sphere $S^2 = \mathbb{C}P^1$. The map

$$\pi : S^3 \rightarrow S^2, \quad \pi(z_1, z_2) = [z_1 : z_2]$$

is a fiber bundle. Explicitly, $\mathbb{C}P^1$ is covered by two charts, $U = \{[z : 1]\} \simeq \mathbb{C}$ and $U' = [1 : z'] \simeq \mathbb{C}$. We have

$$\pi^{-1}(U) = \left\{ \frac{ze^{i\theta}}{\sqrt{1+|z|^2}}, \frac{e^{i\theta}}{\sqrt{1+|z|^2}} \right\} \simeq U \times S^1,$$

and the case of U' is similar.

What can be extracted from (31.1)? Using (31.2) we deduce that

$$\pi_n(S^3) \simeq \pi_n(S^2), \quad n \geq 3,$$

and in particular $\pi_3(S^2) \simeq \pi_3(S^3) \simeq \mathbb{Z}$. It follows that the generator in $\pi_3(S^2)$ is given by the Hopf map $S^3 \rightarrow S^2$. Notice that groups $\pi_n(S^2)$ for $n > 3$ are not known in general, although a great deal of information is available, for example these groups are always finite.

§33. LERAY SPECTRAL SEQUENCE FOR DE RHAM COHOMOLOGY. MAR 14.

We fix a fiber bundle $\pi : E \rightarrow B$ of smooth manifolds (with connected B) and explore how to compute de Rham cohomology of the total space E in terms of de Rham cohomology of the base and the fiber. To apply the spectral sequence of the filtered complex, we need a filtration on de Rham complex

$$A^\bullet = \Omega_{dR}^\bullet(E, \mathbb{R}).$$

It can be defined as follows:

$$F^p A^n = \pi^* \Omega^p(B, \mathbb{R}) \wedge \Omega^{n-p}(E, \mathbb{R}),$$

$$A^n = F^0 A^n \supset F^1 A^n \supset \dots$$

Since de Rham differential commutes with pullback, $F^p A^\bullet$ is indeed a subcomplex of A^\bullet . So from Theorem 27.1, we have a spectral sequence with

$$E_{p,q}^0 = \frac{F^p A^{p+q}}{F^{p+1} A^{p+q}}$$

which abuts to $H_{dR}^\bullet(E, \mathbb{R})$. It is called the *Leray spectral sequence*. Let's explore it in some detail. We have

$$F^p A^{p+q} = \pi^* \Omega^p(B, \mathbb{R}) \wedge \Omega^q(E, \mathbb{R}),$$

Choose a trivializing neighborhood $U \subset B$ of a point $b \in B$ with coordinates x_i , and a trivialization $\pi^{-1}(U) \simeq U \times F_b$ (where $F_b = \pi^{-1}(b)$). Of course F_b is diffeomorphic to F , but there is no canonical choice for this diffeomorphism. Choose a coordinate neighborhood in F_b with coordinates y_i . Then $\omega \in F^p A^{p+q}$ can be locally written as

$$\omega = \sum_{\substack{|I|+|J|=p+q \\ |I| \geq p}} a_{I,J} dy_J \wedge dx_I,$$

where $a_{I,J}$ are some functions. So a class $[\omega] \in E_{p,q}^0$ can be locally written as

$$\omega = \sum_{|I|=p, |J|=q} a_{I,J} dy_J \wedge dx_I.$$

Recall that d_0 is induced by d :

$$d_0[\omega] = [d\omega] = \left[\sum_{|I|=p} d_y \left(\sum_{|J|=q} a_{I,J} dy_J \right) \wedge dx_I \right],$$

where d_y is a "partial" differential in direction of y_j 's only (since partial derivatives in the direction of x_i will give an element of $F^{p+1} A^{p+q}$).

33.1. PROPOSITION. $H_{dR}^\bullet(F, \mathbb{R})$ is finite-dimensional.

We will prove this later, after we discuss some sheaf theory. Given the proposition, we see that elements of $E_1^{p,q}$ are (locally and after the choice of the trivialization) represented by classes

$$\sum_s [\eta_s] \otimes \alpha_s \in H_{dR}^q(F_b, \mathbb{R}) \otimes A^p(B),$$

where η_s are closed forms on F_b such that their classes form a basis of $H_{dR}^q(F_b, \mathbb{R})$. Indeed, any local form as above can be extended to E by multiplying it by a bump function that vanishes outside of some open set $V \supset U$. To go further, we have to analyze how $H_{dR}^q(F_b, \mathbb{R})$ depends on $b \in B$.

§34. MONODROMY. MAR 16

Take $b, b' \in B$ and take any path γ connecting b and b' . We are going to define a "transfer" linear operator $T_\gamma : H_{dR}^q(F_b, \mathbb{R}) \rightarrow H_{dR}^q(F_{b'}, \mathbb{R})$. First, choose trivializing open sets for π along γ . We will define transfer operators first for portions of γ inside one trivializing neighborhood, and then take their composition. Assuming $b, b' \in U$, choose a trivialization $\pi^{-1}(U) \simeq U \times F_b$ such that $\eta|_{F_b} = \text{Id}_{F_b}$. Then $\eta^{-1}|_{F_{b'}}$ gives a diffeomorphism $F_b \rightarrow F_{b'}$, which induces an isomorphism $H_{dR}^q(F_b, \mathbb{R}) \rightarrow H_{dR}^q(F_{b'}, \mathbb{R})$.

34.1. PROPOSITION. *The construction of T_γ does not depend on any choices, in fact it depends only on the homotopy class of γ .*

Proof. To show that T_γ is well-defined, it suffices to show that a local construction above does not depend on a choice of a trivialization. Another trivialization η' will give another diffeomorphism $F_b \rightarrow F_{b'}$, but this diffeomorphism will be homotopic to the one constructed above. Indeed, a composition $\eta^{-1}\eta'$ gives a diffeomorphism $F_b \rightarrow F_b$ homotopic to the identity map (the homotopy is obtained by moving from b' to b along γ). Homotopic maps induce the same maps on cohomology. The same reasoning shows that T_γ depends only on the homotopy class of γ . \square

34.2. DEFINITION. Choose a base point $b \in B$. A map $\pi_1(B) \rightarrow \text{Aut}(H_{dR}^q(F, \mathbb{R}))$ given by $\gamma \rightarrow T_\gamma$ is called a monodromy homomorphism. (Notice that we can also use singular cohomology with integer coefficients to define monodromy).

If monodromy is trivial then we can canonically identify cohomology groups of all fibers. In this case the Leray spectral sequence takes a particularly simple form. Indeed, we see that in this case we have

$$E_1^{p,q} = H_{dR}^q(F, \mathbb{R}) \otimes A^p(B),$$

and d_1 will be a de Rham differential from B . So we have

34.3. THEOREM. *In the absence of monodromy (for example if $\pi_1(B) = \{e\}$), the Leray spectral sequence abuts to $H_{dR}^{p+q}(E, \mathbb{R})$ and has*

$$E_2^{p,q} = H_{dR}^q(F, \mathbb{R}) \otimes H_{dR}^p(B, \mathbb{R}). \tag{34.4}$$

§35. LERAY–HIRSH THEOREM. KÜNNETH FORMULA. MAR 26

35.1. THEOREM. *Let $\pi : E \rightarrow B$ be a fiber bundle with fiber $F \hookrightarrow E$ and suppose the inclusion of the fiber induces a surjection of cohomology groups $H_{dR}^k(E, \mathbb{R}) \rightarrow H_{dR}^k(F, \mathbb{R})$. Then we have (not canonically),*

$$H_{dR}^n(E, \mathbb{R}) = \bigoplus_{p+q=n} H_{dR}^q(F, \mathbb{R}) \otimes H_{dR}^p(B, \mathbb{R}).$$

35.2. COROLLARY (Künneth formula).

$$H_{dR}^n(X \times Y, \mathbb{R}) = \bigoplus_{p+q=n} H_{dR}^q(X, \mathbb{R}) \otimes H_{dR}^p(Y, \mathbb{R}).$$

Proof. In the Leray–Hirsch set-up, consider the Leray spectral sequence. We claim that the monodromy is trivial, and therefore we have (34.4), and moreover $d_2 = d_3 = \dots = 0$, and so $E_\infty = E_2$. We fix a graded subspace $V \subset H_{dR}^\bullet(E, \mathbb{R})$ that maps isomorphically onto $H_{dR}^\bullet(F, \mathbb{R})$. Fix a path γ connecting points $b, b' \in B$. We claim that we have a commutative diagram

$$\begin{array}{ccc} V & \xlongequal{\quad} & V \\ \downarrow & & \downarrow \\ H_{dR}^\bullet(F_b, \mathbb{R}) & \xrightarrow{T_\gamma} & H_{dR}^\bullet(F_{b'}, \mathbb{R}) \end{array}$$

where the vertical maps are induced by inclusions of these fibers into E . In particular, transfer maps do not depend on γ , i.e. monodromy is trivial. It

suffices to prove this under assumption that b, b' belong to the same trivializing neighborhood. So we can assume that $E \simeq B \times F$. The transfer map $H_{dR}^\bullet(F_b, \mathbb{R}) \xrightarrow{T_\gamma} H_{dR}^\bullet(F_{b'}, \mathbb{R})$ is then induced by a diffeomorphism $F_b \simeq F_{b'}$ given by the projection $B \times F \rightarrow F$. Since maps $F_b \hookrightarrow E$ and $F_{b'} \hookrightarrow E$ are homotopic, they induce the same pull-back map on cohomology.

It remains to show that all higher differentials in the Leray spectral sequence vanish. Take a class in $E_2^{p,q} = H_{dR}^q(F, \mathbb{R}) \otimes H_{dR}^p(B, \mathbb{R})$. It is represented by a differential form $\alpha = \sum_i \eta_i \wedge \pi^*(\omega_i) \in F^p A^{p+q}$, where $\eta_i \in V^q$ and $\omega_i \in \Omega^p(B)$ are closed forms. Therefore, α is closed and so $d_2(\alpha) = d_3(\alpha) = \dots = 0$. \square

§36. CALCULATION FOR THE HOPF BUNDLE.

Consider again the Hodge fibration $S^3 \rightarrow S^2$ with fiber S^1 . Since S^2 is simply-connected, the E_2 page of the Leray spectral sequence is

$$\begin{array}{ccc} \mathbb{R} & 0 & \mathbb{R} \\ \mathbb{R} & 0 & \mathbb{R} \end{array}$$

Since S^3 has trivial cohomology in degrees 1 and 2, we see that

$$d_2 : \mathbb{R} \rightarrow \mathbb{R}$$

(from the top left to the bottom right corner) is an isomorphism. Explicitly, $H_{dR}^1(S^1, \mathbb{R}) \simeq \mathbb{R}$ is generated by an angle form $\omega = \frac{d\theta}{2\pi}$ (here $S^1 = \{e^{i\theta}\}$). Where is it sent by d_2 ? Notice that ω is the restriction of the form

$$\alpha = \frac{i}{2\pi}(z_1 d\bar{z}_1 + z_2 d\bar{z}_2)$$

on each fiber

$$\{z_1 = e^{i\theta} z_1^0, \quad z_2 = e^{i\theta} z_2^0\} \subset S^3 \subset \mathbb{C}_{z_1, z_2}^2.$$

It follows that $d_2(\omega) = d\alpha$ (more precisely, $d_2(\omega)$ is a form on S^2 such that its pull-back is $d\alpha$). So we compute

$$d\alpha = \frac{i}{2\pi}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2).$$

It is often desirable to have a form on $\mathbb{C}^2 \setminus \{0\}$, which is a pull-back of a form on $\mathbb{C}\mathbb{P}^1$ with respect to the map $\mathbb{C}^2 \setminus \{0\} \rightarrow \mathbb{C}\mathbb{P}^1, (z_1, z_2) \mapsto [z_1 : z_2]$. The standard choice is

$$\alpha = \frac{i}{2\pi} \frac{z_1 d\bar{z}_1 + z_2 d\bar{z}_2}{|z_1|^2 + |z_2|^2} = \frac{i}{2\pi} \bar{\partial} \log(|z_1|^2 + |z_2|^2)$$

and

$$d\alpha = \frac{i}{2\pi} \partial \bar{\partial} \log(|z_1|^2 + |z_2|^2).$$

Notice that α does not change if we multiply z_1, z_2 by the same multiple. So it is a pull-back of the form on $S^2 \simeq \mathbb{C}\mathbb{P}^1$.

§37. SHEAVES. RINGED SPACES. MARCH 28.

37.1. DEFINITION. Let X be a topological space. A category $\mathbf{Top}(X)$ has open subsets of X as objects and inclusions of opens sets $U \hookrightarrow V$ as morphisms. A *presheaf* \mathcal{F} of abelian groups is a contravariant functor $\mathbf{Top}(X) \rightarrow \mathbf{Ab}$. This includes the following data:

- an abelian group $\mathcal{F}(U)$ (called the group of *local sections*) for each open subset $U \subset X$;
- A homomorphism $\mathcal{F}(V) \rightarrow \mathcal{F}(U)$ for each inclusion of opens sets $U \hookrightarrow V$. This homomorphism is called *restriction* and is typically denoted by $s \mapsto s|_U$.

The group $\Gamma(X, \mathcal{F}) := \mathcal{F}(X)$ is called the group of *global sections*.

37.2. DEFINITION. A presheaf is called a *sheaf* if it satisfies two additional axioms. For any open subset U and its covering $U = \cup_{\alpha} U_{\alpha}$, we have

- If $s \in \mathcal{F}(U)$ and $s|_{U_{\alpha}} = 0$ for any α then $s = 0$.
- Given a collection $(s_{\alpha}) \in \prod_{\alpha} \mathcal{F}(U_{\alpha})$ such that

$$s_{\alpha}|_{U_{\alpha} \cap U_{\beta}} = s_{\beta}|_{U_{\alpha} \cap U_{\beta}}$$

for any α, β , there exists $s \in \mathcal{F}(U)$ such that $s_{\alpha} = s|_{U_{\alpha}}$ for any α .

In other words, sections of a sheaf can be glued from compatible local data.

37.3. REMARK. We can define sheaves of rings, sets, etc. in the same way, by changing the target category in the functor $\mathbf{Top}(X) \rightarrow \mathbf{Ab}$.

37.4. REMARK. There are two main sources of sheaves: topology and geometry. In topology, the main example is a constant sheaf. In geometry, the main example is a sheaf of sections of a vector bundle. Using various operations on sheaves (cokernels of morphisms of sheaves, push-forward of a sheaf by a continuous map, etc.) one quickly builds interesting categories of sheaves, such as constructible sheaves in topology or coherent sheaves in complex (or algebraic) geometry.

37.5. EXAMPLE. Let A be an abelian group, e.g. \mathbb{Z} . A *constant presheaf* of stalk A has $\mathcal{F}(U) = A$ for any open U and all restriction homomorphisms are identity maps. This is not a sheaf because a gluing axiom will be violated for disconnected open sets U . To fix this issue, we define a *constant sheaf* of stalk A as follows: $\mathcal{F}(U)$ is the set of all locally constant maps from U to A , i.e. all continuous maps $U \rightarrow A$ if A is endowed with discrete topology.

37.6. DEFINITION. In geometry, X is typically not just a topological space but has some distinguished class of functions on it. This is formalized as follows: a *ringed space* is a topological space with a sheaf of rings on it. This sheaf is often called the *structure sheaf*.

37.7. EXAMPLE. For example, any topological space carries a sheaf \mathcal{C}^0 of continuous functions: local sections $\mathcal{C}^0(U)$ are just continuous functions $U \rightarrow R$, with obvious restrictions $\mathcal{C}^0(U) \rightarrow \mathcal{C}^0(V)$ for any inclusion $V \hookrightarrow U$. A smooth manifold X carries a sheaf \mathcal{C}^{∞} of smooth functions. A complex manifold X carries a sheaf \mathcal{O}_X of holomorphic functions. An algebraic variety (or scheme) X carries a sheaf \mathcal{O}_X of regular functions.

37.8. DEFINITION. Let X be a space with a sheaf of rings \mathcal{A} . A *sheaf of \mathcal{A} -modules* is a sheaf of abelian groups \mathcal{F} with the following extra data: for any open subset $U \subset X$, $\mathcal{F}(U)$ is an $\mathcal{A}(U)$ -module, and we have a commutative diagram

$$\begin{array}{ccc} \mathcal{A}(U) \times \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{A}(V) \times \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V) \end{array}$$

for any inclusion $V \hookrightarrow U$, where horizontal maps are action maps of the ring on the module and vertical maps are restriction maps.

For example, if X is a manifold then \mathcal{C}^∞ is obviously a sheaf of \mathcal{C}^∞ -modules, but the constant sheaf \mathbb{Z} is not: a product of a smooth function and a locally-constant function is typically not locally-constant.

37.9. DEFINITION. A *direct sum of sheaves* (or sheaves of \mathcal{A} -modules) is defined as follows: a group of local sections of $\oplus \mathcal{F}_i$ on U is a direct sum of groups of local sections $\oplus \mathcal{F}_i(U)$, with component-wise restriction maps.

37.10. DEFINITION. Sheaves of abelian groups on X form a category, denoted by \mathbf{Sh}/X . Morphisms of sheaves $f : \mathcal{F} \rightarrow \mathcal{G}$ are natural transformations of the corresponding functors $\mathbf{Top}(X) \rightarrow \mathbf{Ab}$. More concretely, for any open subset $U \subset X$ we need a homomorphism $f|_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ such that we have a commutative diagram of restrictions

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f|_U} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \xrightarrow{f|_V} & \mathcal{G}(V) \end{array}$$

for any open inclusion $V \subset U$.

37.11. DEFINITION. Let $U \subset X$ be an open subset and let \mathcal{F} be a sheaf on X . Its restriction $\mathcal{F}|_U$ is a sheaf on U defined as follows: $\mathcal{F}|_U(V) = \mathcal{F}(V)$ for any open subset $V \subset U$. If \mathcal{A} is a sheaf of rings on X and \mathcal{F} is a sheaf of \mathcal{A} -modules then clearly $\mathcal{F}|_U$ is a sheaf of $\mathcal{A}|_U$ -modules. One should not confuse $\mathcal{F}|_U$ and $\mathcal{F}(U)$: the first object is a sheaf, and includes information about all groups of local sections on subsets $V \subset U$, while the second object is just the group of local sections on U .

37.12. DEFINITION. Let $f : X \rightarrow Y$ be a continuous map of topological spaces and let \mathcal{F} be a sheaf on X . Its pushforward $f_*\mathcal{F}$ is a sheaf on Y defined as follows: $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$ for any open $U \subset Y$, with restriction maps $f_*\mathcal{F}(U) \rightarrow f_*\mathcal{F}(V)$ given by restriction maps $\mathcal{F}(f^{-1}(U)) \rightarrow \mathcal{F}(f^{-1}(V))$ for any $V \subset U$.

37.13. DEFINITION. Let (X, \mathcal{A}) be a ringed space. A sheaf of \mathcal{A} -modules \mathcal{F} is called *free* of rank n if it is isomorphic to \mathcal{A}^n . A sheaf of \mathcal{A} -modules is called *locally free* of rank n if any point has a neighborhood U such that $\mathcal{F}|_U$ is isomorphic to $\mathcal{A}^n|_U$.

37.14. THEOREM. Let X be a complex manifold. Then there is a natural bijection (described in the proof) between the set of isomorphism classes of locally free \mathcal{O}_X -modules of rank n and the set of isomorphism classes of vector bundles on X of rank n . The same result holds for smooth manifolds (with C^∞ instead of \mathcal{O}_X).

Proof. Let E be a vector bundle on X . This means that E is a complex manifold with a holomorphic map $\pi : E \rightarrow X$ such that X has a covering $X = \cup U_i$ with the following properties. We have biholomorphic isomorphisms $\tau_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{C}^n$, called *trivializations*, such that $\text{pr}_1 \circ \tau_i = \pi|_{\pi^{-1}(U_i)}$. The functions $\tau_j \circ \tau_i^{-1}$, defined on overlaps $U_i \cap U_j$, have the form

$$(x, v) \mapsto (x, g_{ij}(x)v),$$

where

$$g_{ij} : U_i \cap U_j \rightarrow \text{GL}_n(\mathbb{C})$$

are called *transition functions*. We clearly have a *cocycle condition*

$$g_{ki} \circ g_{jk} \circ g_{ij} = \tau_i \circ \tau_k^{-1} \circ \tau_k \circ \tau_j^{-1} \circ \tau_j \circ \tau_i^{-1} = 1$$

on each triple overlap $U_i \cap U_j \cap U_k$. It is straightforward to show that any choice of transition functions satisfying the cocycle condition produces a vector bundle. Namely, one glues E from open charts $U_i \times \mathbb{C}^n$ using transition functions to glue charts together.

The corresponding sheaf \mathcal{E} (called the *sheaf of sections of a vector bundle*) is defined as follows: for an open subset $U \subset X$, $\mathcal{E}(U)$ is a set of sections over U , i.e. maps $s : U \rightarrow E$ such that $\pi \circ s = \text{Id}_U$. We can add sections (point-wise) and multiply them by holomorphic functions in $\mathcal{O}_X(U)$ (also point-wise), so \mathcal{E} is a sheaf of \mathcal{O}_X -modules. If $U \subset X$ is a trivializing neighborhood then a section $s \in \mathcal{E}(V)$ for $V \subset U$ is determined by n holomorphic functions s_1, \dots, s_n on V , so \mathcal{E} is locally-free.

Given a locally free sheaf \mathcal{E} , we can construct a vector bundle E as follows. Choose an atlas $\cup U_i$ such that $\mathcal{E}|_{U_i} \simeq \mathcal{O}_X^n|_{U_i}$, call this isomorphism τ_i . This will be a trivializing atlas for a vector bundle. Transition functions are obtained by computing $\tau_j \circ \tau_i^{-1}$ on overlaps. Indeed, an isomorphism

$$G_{ij} : \mathcal{O}_X^n|_{U_i \cap U_j} \rightarrow \mathcal{O}_X^n|_{U_i \cap U_j}$$

is given by a compatible collection of isomorphisms

$$\mathcal{O}_X(V)^n \rightarrow \mathcal{O}_X(V)^n$$

for all open subsets $V \subset U_i \cap U_j$, but those are completely determined by images of sections $(0, \dots, 0, 1, 0, \dots, 0)$. So G_{ij} is completely determined by an isomorphism

$$\mathcal{O}_X(U_i \cap U_j)^n \rightarrow \mathcal{O}_X(U_i \cap U_j)^n,$$

which is given by an invertible $n \times n$ matrix of holomorphic functions. \square

37.15. EXAMPLE. Let X be a manifold. Then we have sheaves Ω^k of differential k -forms: $\Omega^k(U)$ is a group of k -forms on U , with obvious restriction maps. Ω^k is a sheaf of C^∞ -modules. It can also be defined as a sheaf of sections of the k -th exterior power of the cotangent bundle. We have a sheaf version of the de Rham complex

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \dots$$

where maps d (de Rham differentials) are morphisms of sheaves of abelian groups (but not morphisms of sheaves of C^∞ -modules because of the Leibniz rule: $d(f\omega) \neq fd(\omega)$ for $f \in C^\infty(U)$, $\omega \in \Omega^k(U)$). The usual de Rham complex is nothing but the induced complex of global sections

$$\Gamma(X, \Omega^0) \xrightarrow{d} \Gamma(X, \Omega^1) \xrightarrow{d} \Gamma(X, \Omega^2) \xrightarrow{d} \dots$$

If X is a complex manifold then Ω^k typically denotes the sheaf of holomorphic k -forms. This is a sheaf of \mathcal{O}_X -modules. We also have sheaves of (complexified) smooth k -forms \mathcal{A}^k , which are direct sums $\mathcal{A}^k = \bigoplus_{p+q=k} \mathcal{A}^{p,q}$ of Dolbeaut sheaves of k -forms of type (p, q) . \mathcal{A}^k and $\mathcal{A}^{p,q}$ are sheaves of \mathcal{A}^0 -modules, where $\mathcal{A}^0 = C^\infty \otimes \mathbb{C}$ is a sheaf of complex-valued smooth functions.

§38. STALKS. SHEAFIFICATION. MARCH 30

38.1. DEFINITION. Let \mathcal{F} be a presheaf on X , $x \in X$. A *stalk* \mathcal{F}_x is a direct limit of groups $\mathcal{F}(U)$ for $x \in U$ partially ordered by inclusion.

Categorically, we have homomorphisms $\mathcal{F}(U) \rightarrow \mathcal{F}_x$ for any neighborhood $x \in U$ which commute with restriction maps $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$ for $V \subset U$. Moreover, \mathcal{F}_x is universal with respect to this property.

Concretely, elements of \mathcal{F}_x are represented by pairs $[U, s]$, where U is a neighborhood of x and $s \in \mathcal{F}(U)$. Two such pairs $[U, s]$, $[U', s']$ are equivalent if there exists a neighborhood $V \subset U \cap U'$ such that $s|_V = s'|_V$. A zero element of \mathcal{F}_x is represented by any pair $[U, 0]$. An element of \mathcal{F}_x represented by $[U, s]$ is called a *germ* of s and is denoted by s_x .

38.2. EXAMPLE. A stalk of a constant sheaf \mathbb{Z} at $x \in X$ is \mathbb{Z} : a germ of a section in $\mathbb{Z}(U)$, i.e. a locally constant function $f : U \rightarrow \mathbb{Z}$, is $f(x)$.

A stalk of a structure sheaf \mathcal{O}_X of an n -dimensional complex manifold \mathcal{O}_X at any point is isomorphic to the ring of convergent power series in n variables. Indeed, a holomorphic function in n variables is determined by its power series expansion. Notice that this ring is local: any convergent power series with non-vanishing constant term is invertible. So X is an example of a *locally ringed space*: a ringed space such that stalks of the structure sheaf are local rings.

38.3. DEFINITION. Let \mathcal{F} be a presheaf on X . Its sheafification \mathcal{F}^+ is a sheaf along with a morphism of presheaves $\mathcal{F} \rightarrow \mathcal{F}^+$ such that any morphism $\mathcal{F} \rightarrow \mathcal{G}$ to a sheaf \mathcal{G} uniquely factors through \mathcal{F}^+ .

The presheaf is constructed as follows:

$$\mathcal{F}^+(U) = \left\{ s : U \rightarrow \prod_{x \in X} \mathcal{F}_x \mid s(x) \text{ is locally given by a section of } \mathcal{F} \right\}.$$

This means that for any $x \in U$, $s(x) \in \mathcal{F}_x$ and there exists a neighborhood V of x and a section $t \in \mathcal{F}(V)$ such that $s(y) = t_y$ for any $y \in V$. More concretely, if we choose such a neighborhood for any point of U , we can interpret elements of $\mathcal{F}^+(U)$ as follows: there exists a covering $U = \cup_{i \in I} U_i$ and sections $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ on any overlap.

One can show that

- (1) \mathcal{F}^+ is a presheaf;
- (2) \mathcal{F}^+ is a sheaf;
- (3) we have a natural morphism $\mathcal{F} \rightarrow \mathcal{F}^+$;
- (4) this morphism satisfies the universal property of the sheafification.

§39. ABELIAN CATEGORIES. EXACT SEQUENCES OF SHEAVES. APRIL 2

We are going to show that the category of sheaves on a topological space is an abelian category, so it's time to give a rigorous definition.

39.1. DEFINITION. A category is called *additive* if $\text{Hom}(X, Y)$ is an abelian group for any objects X, Y and the composition law $\text{Hom}(X, Y) \times \text{Hom}(Y, Z) \rightarrow \text{Hom}(X, Z)$ is bilinear. Functor of additive categories $\phi : C \rightarrow C'$ is an additive functor, i.e. $\text{Hom}(X, Y) \xrightarrow{\phi} \text{Hom}(\phi(X), \phi(Y))$ is a homomorphism for any objects X, Y .

39.2. DEFINITION. An additive category is called *abelian* if the following axioms are satisfied:

- (1) Any morphism $f : X \rightarrow Y$ has a kernel, i.e. an object $\text{Ker } f$ and a morphism $\text{Ker } f \rightarrow X$ which satisfies the universal property: any morphism $M \rightarrow X$ such that its composition with $X \rightarrow Y$ is a zero map, uniquely factors as $M \rightarrow \text{Ker } f \rightarrow X$.
- (2) Any morphism $f : X \rightarrow Y$ has a cokernel, i.e. an object $\text{Coker } f$ and a morphism $Y \rightarrow \text{Coker } f$ which satisfies the universal property: any morphism $Y \rightarrow N$ such that its composition with $X \rightarrow Y$ is a zero map, uniquely factors as $Y \rightarrow \text{Coker } f \rightarrow N$.
- (3) Finite direct sums exist.
- (4) Any morphism $f : X \rightarrow Y$ has a unique (up to an isomorphism) factorization $X \rightarrow I \rightarrow Y$, such that I is both a cokernel of $\text{Ker } f \rightarrow X$ and a kernel of $Y \rightarrow \text{Coker } f$. An object I is called the image of f .

39.3. REMARK. In undergraduate algebra, one typically defines the image of the homomorphism of abelian groups first, then the quotient group, and then the cokernel as the quotient of Y by the image. The last axiom then becomes the first isomorphism theorem.

39.4. REMARK. In general one can't think about morphisms of objects as functions of sets. So diagram-chasing is not directly applicable. However, it can still be used, either by using "Grothendieck points" or by invoking the Freyd–Mitchell embedding theorem, which says that any abelian category is isomorphic to a full subcategory of some $R\text{-mod}$. So any statement proved for arbitrary R -modules using nothing but diagram chasing, e.g. the snake lemma or the 5-lemma, is valid in any abelian category.

Let's show that \mathbf{Sh}_X is an Abelian category. Basically, we have to define the kernel, the cokernel, and the image of any morphism of sheaves $f : \mathcal{F} \rightarrow \mathcal{G}$. In the category of presheaves, we can define presheaves

$$\begin{aligned} \widetilde{\text{Ker}}(U) &= \text{Ker}[\mathcal{F}(U) \rightarrow \mathcal{G}(U)], \\ \widetilde{\text{Coker}}(U) &= \text{Coker}[\mathcal{F}(U) \rightarrow \mathcal{G}(U)], \\ \widetilde{\text{Im}}(U) &= \text{Im}[\mathcal{F}(U) \rightarrow \mathcal{G}(U)]. \end{aligned}$$

It turns out that if \mathcal{F} and \mathcal{G} are sheaves then $\widetilde{\text{Ker}} f$ is automatically a sheaf, and so we set $\text{Ker } f = \widetilde{\text{Ker}} f$, but $\text{Coker } f$ and $\text{Im } f$ have to be defined as sheafifications of $\widetilde{\text{Coker}} f$ and $\widetilde{\text{Im}} f$. Nevertheless, it is useful to remember that stalks of the image and the cokernel sheaves can be computed as stalks of their presheaves analogues.

Details are left as exercises, along with most of the proof of the following extremely useful fact:

39.5. LEMMA. *A complex of sheaves $\mathcal{F}^i \rightarrow \mathcal{F}^{i+1} \rightarrow \dots$ is exact if and only if an induced complex of stalks $\mathcal{F}_x^i \rightarrow \mathcal{F}_x^{i+1} \rightarrow \dots$ is exact for any $x \in X$.*

We will only prove the main point: a morphism of sheaves $f : \mathcal{F} \rightarrow \mathcal{G}$ is surjective if it is surjective on stalks. So suppose $f_x(\mathcal{F}_x) = \mathcal{G}_x$ for any $x \in X$. Take an open subset $U \subset X$ and a section $s \in \mathcal{G}(U)$. We claim that s is a section of the image sheaf. Indeed, for any point $x \in U$ there exists an element of \mathcal{F}_x that maps to s_x . Choosing its representative, we get a neighborhood V of x and a section $t \in \mathcal{F}(V)$ such that $s_x = f_x(t_x)$. Therefore s and $f(t)$ have the same germ at x . So we can find a smaller neighborhood W such that $s|_W = f(t)|_W$, i.e. $s|_W$ is an image of a local section of \mathcal{F} . By definition of sheafification, this implies that s belongs to the image sheaf $\text{Im}(f)$.

§40. COHOMOLOGY OF SHEAVES. APRIL 4.

Let X be a topological space. We have a functor of global sections

$$\mathbf{Sh}_X \rightarrow \mathbf{Ab}, \quad \mathcal{F} \mapsto \Gamma(X, \mathcal{F}).$$

40.1. DEFINITION. The k -th cohomology group $H^k(X, \mathcal{F})$ is the k -th right-derived functor $R^k\Gamma$ of the functor of global sections.

To make sense of this definition we have to prove that

40.2. LEMMA. $\Gamma(X, \cdot)$ is left-exact.

and

40.3. LEMMA. \mathbf{Sh}_X has enough injectives.

Proof. We followed Voisin's textbook. □

40.4. REMARK. In practice one computes $H^k(X, \mathcal{F})$ using the δ -functor formalism, i.e. by converting short exact sequences of sheaves into long exact sequences of cohomology groups and by using functoriality of connecting homomorphisms to deal with more complicated situations. However, often we need to get a concrete description of cohomology groups. Using injective resolutions is impractical.

40.5. DEFINITION. An exact sequence of sheaves

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{L}_0 \rightarrow \mathcal{L}_1 \rightarrow \dots \quad (40.6)$$

is called an *acyclic* resolution if all sheaves \mathcal{L}_k are acyclic, i.e. $H^i(X, \mathcal{L}_k) = 0$ for $i > 0$.

40.7. LEMMA. Given an acyclic resolution (40.6), $H^k(X, \mathcal{F})$ is isomorphic to the k -th cohomology group of the complex of global sections

$$0 \rightarrow \Gamma(X, \mathcal{L}_0) \rightarrow \Gamma(X, \mathcal{L}_1) \rightarrow \dots$$

Proof. We gave two simple proofs. □

Which sheaves are acyclic?

40.8. DEFINITION. A sheaf \mathcal{F} is called *flasque* if the restriction homomorphism $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$ is surjective for any inclusion $V \subset U$.

40.9. LEMMA. *Flasque sheaves are acyclic.*

Proof. We followed Voisin’s textbook. □

40.10. EXAMPLE. For any sheaf \mathcal{F} , consider the Godement sheaf \mathcal{F}_{God} with

$$\mathcal{F}_{God}(U) = \prod_{x \in U} \mathcal{F}_x$$

and obvious restriction maps. This sheaf is obviously flasque. Moreover, the map sending a local section of \mathcal{F} into a collection of its germs at all points gives a morphism $\mathcal{F} \rightarrow \mathcal{F}_{God}$, which in fact is injective, because a local section of a sheaf vanishes if and only if all its germs vanish. This gives a flasque Godement resolution

$$0 \rightarrow \mathcal{F} \xrightarrow{f} \mathcal{F}_{God} \rightarrow (\text{Im } f)_{God} \rightarrow \dots,$$

which has a nice feature of being functorial in \mathcal{F} .

40.11. DEFINITION. A sheaf \mathcal{F} is called *fine* if \mathcal{F} is a sheaf of \mathcal{A} -modules on a ringed space and \mathcal{A} has a partition of unity property: for any covering $X = \cup_i U_i$, there exist sections $\rho_i \in \Gamma(X, \mathcal{A})$ such that

- $(\rho_i)_x = 0$ for any $x \notin U_i$;
- for any $x \in X$, only finitely many germs $(\rho_i)_x$ are not equal to zero;
- $\sum \rho_i = 1$.

40.12. LEMMA. *Fine sheaves are acyclic.*

Proof. We followed Voisin’s textbook. □

This is already very useful. For example, let X be a manifold. Then any sheaf of C^∞ -modules, for example a sheaf of sections of any smooth vector bundle is a fine sheaf. In particular, sheaves Ω^k are fine.

§41. DE RHAM RESOLUTION. DOLBEAUT RESOLUTION. APRIL 6.

Let X be a smooth manifold. Consider a sheaf-theoretic de Rham complex

$$0 \rightarrow \mathbb{R} \rightarrow \Omega^0 \rightarrow \Omega^1 \rightarrow \Omega^2 \rightarrow \dots$$

By Poincare Lemma, any closed form is locally exact, i.e. this complex is in fact a resolution of the constant sheaf \mathbb{R} . By Lemma 40.12, all sheaves Ω^k are acyclic on X . By Lemma 40.7, we have

41.1. THEOREM. $H^k(X, \mathbb{R})$ is isomorphic to the k -th cohomology group of the complex

$$0 \rightarrow \Gamma(X, \Omega^0) \rightarrow \Gamma(X, \Omega^1) \rightarrow \Gamma(X, \Omega^2) \rightarrow \dots,$$

i.e. to the de Rham cohomology group $H_{dR}^k(X, \mathbb{R})$.

A similar analysis, based on the theorem of small chains instead of Poincaré Lemma, gives

41.2. THEOREM. Let A be an abelian group (e.g. \mathbb{Z}). $H^k(X, A)$ is isomorphic to the k -th singular cohomology group $H_{sing}^k(X, A)$.

We skipped the proof.

Now let X be a complex manifold with a sheaf \mathcal{O}_X of holomorphic functions. A $\bar{\partial}$ -Poincaré Lemma gives a resolution

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{A}^{0,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{0,1} \xrightarrow{\bar{\partial}} \mathcal{A}^{0,2} \rightarrow \dots$$

Again all sheaves $\mathcal{A}^{0,k}$ are fine sheaves of $\mathcal{A}^{0,0}$ -modules (recall that $\mathcal{A}^{0,0}$ is a sheaf of complex-valued smooth functions). In particular, we get

41.3. THEOREM. $H^q(X, \mathcal{O}_X)$ is isomorphic to the q -th cohomology group of the complex

$$0 \rightarrow \mathcal{A}^{0,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{0,1} \xrightarrow{\bar{\partial}} \mathcal{A}^{0,2} \rightarrow \dots,$$

i.e. to the Dolbeault cohomology group $H^{0,q}$, see Section §15.

More generally, we can consider an analogous resolution of the sheaf of holomorphic p -forms (and in fact of the sheaf of holomorphic sections of any holomorphic vector bundle):

$$0 \rightarrow \Omega^p \rightarrow \mathcal{A}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,2} \rightarrow \dots$$

In particular, we see that $H^q(X, \Omega^p)$ is isomorphic to the q -th cohomology group of the complex

$$0 \rightarrow \mathcal{A}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,2} \rightarrow \dots,$$

i.e. to the Dolbeault cohomology group $H^{p,q}$.

What is the relation between de Rham and Dolbeault cohomology groups? In 15.1 we have already realized $\Omega^k(M, \mathbb{C})$ as a total complex of the double complex $(A^{p,q}, \partial, \bar{\partial})$. So a general formalism of spectral sequences gives a Frölicher spectral sequence

$$E_1^{p,q} = H^{p,q} \quad \Rightarrow \quad H^{p+q}(X, \mathbb{C}).$$

A remarkable fact is that this spectral sequence converges right away in the most important case:

41.4. THEOREM. Suppose X is a projective algebraic variety or a compact Kähler manifold. Then the Frölicher spectral sequence degenerates at E_1 , and moreover we have a canonical Hodge decomposition

$$\bigoplus_{p+q=n} H^{p,q} \simeq H^n(X, \mathbb{C}).$$

We discussed the shape of the Hodge diamond of algebraic curves and algebraic surfaces.

§42. ČECH COMPLEX. APRIL 9 AND 11.

The most “hands-on” way of computing cohomology is undoubtedly given by the Čech complex. Let X be a topological space, let \mathcal{F} be a sheaf, and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be a covering of X by open sets. We assume that I is totally ordered.

42.1. DEFINITION. Čech cohomology groups $H_{\mathcal{U}}^k(X, \mathcal{F})$ are the k -th cohomology groups of the Čech complex

$$0 \rightarrow \bigoplus_{\alpha} \mathcal{F}(U_\alpha) \rightarrow \bigoplus_{\alpha < \beta} \mathcal{F}(U_\alpha \cap U_\beta) \rightarrow \bigoplus_{\alpha < \beta < \gamma} \mathcal{F}(U_\alpha \cap U_\beta \cap U_\gamma) \rightarrow \dots$$

We think about Čech cochains as collections of sections σ_{i_1, \dots, i_k} indexed by ordered k -tuples of indices in I . The differentials are defined as follows:

$$(d\sigma)_{i_0 \dots i_k} = \sigma_{i_1 \dots i_k} - \sigma_{i_0 i_2 \dots i_k} + \dots + (-1)^k \sigma_{i_0 i_1 \dots i_{k-1}}.$$

More precisely, one has to restrict sections on the right to $U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_k}$.

For instance, it is clear that elements of $H_{\mathcal{U}}^0(X, \mathcal{F})$ are given by collections of sections s_α which agree on overlaps, i.e. by global sections of \mathcal{F} . More generally, we have

42.2. THEOREM. If $H^n(U_{i_1} \cap \dots \cap U_{i_k}, \mathcal{F}) = 0$ for any intersection and any $n > 0$ then we have a natural isomorphism

$$H^n(X, \mathcal{F}) \simeq H_{\mathcal{U}}^n(X, \mathcal{F}) \quad \text{for any } n.$$

42.3. REMARK. It is clear that the Čech complex, and therefore Čech cohomology, is functorial in \mathcal{F} .

Proof. We followed Voisin’s textbook. □

§43. INTERPRETATIONS OF H^1 . PICARD GROUP. APRIL 13.

§44. DERIVED CATEGORIES – I. APRIL 18. GUEST LECTURER RINA ANNO. NOTES TYPED BY TOM.

Let $C(\mathcal{A})$ denote the category of differential complexes of objects in an abelian category \mathcal{A} . An object in $C(\mathcal{A})$ is a complex

$$A^\bullet = \dots \longrightarrow A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} \dots$$

with $d^i \circ d^{i-1} = 0$, and a morphism $f : A^\bullet \rightarrow B^\bullet$ is a collection of $f^i \in \text{Mor}(A^i, B^i)$ making the following diagram commute

$$\begin{array}{ccccccc} \dots & \longrightarrow & A^{i-1} & \xrightarrow{d^{i-1}} & A^i & \xrightarrow{d^i} & A^{i+1} \xrightarrow{d^{i+1}} \dots \\ & & \downarrow f^{i-1} & & \downarrow f^i & & \downarrow f^{i+1} \\ \dots & \longrightarrow & B^{i-1} & \xrightarrow{d^{i-1}} & B^i & \xrightarrow{d^i} & B^{i+1} \xrightarrow{d^{i+1}} \dots \end{array}$$

Since $d^i \circ d^{i-1} = 0$, we have that $\text{im } d^{i-1}$ is a subobject of $\ker d^i$. We define the i th **cohomology group** of A^\bullet to be the quotient $H^i(A^\bullet) := \ker d^i / \text{im } d^{i-1}$. A morphism $f : A^\bullet \rightarrow B^\bullet$ induces a well defined map on cohomology $\overline{f^i} : H^i(A^\bullet) \rightarrow H^i(B^\bullet)$.

44.1. DEFINITION. We say that $f : A^\bullet \rightarrow B^\bullet$ is a **quasi-isomorphism** if all $\overline{f^i} : H^i(A^\bullet) \rightarrow H^i(B^\bullet)$ are isomorphisms.

44.2. DEFINITION (Derived Category). The **derived category** of $C(\mathcal{A})$ is a category $\mathcal{D}(\mathcal{A})$ along with a functor $Q : C(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{A})$ satisfying the following:

- 1) If f is a quasi-isomorphism in $C(\mathcal{A})$ then Qf is an isomorphism in $\mathcal{D}(\mathcal{A})$.
- 2) If $R : C(\mathcal{A}) \rightarrow X$ is a functor such that quasi-isomorphisms map to isomorphisms, then there exists a unique functor $S : \mathcal{D}(\mathcal{A}) \rightarrow X$ such that $SQ = R$

Construction of the derived category.

As a first attempt at constructing the derived category $\mathcal{D}(\mathcal{A})$ of the category of complexes $C(\mathcal{A})$, we can just add in formal inverses to all of the quasi-isomorphisms in $\text{Mor}(C(\mathcal{A}))$. That is, we define $\mathcal{D}(\mathcal{A})$ by saying that $\text{Ob}(\mathcal{D}(\mathcal{A})) = \text{Ob}(C(\mathcal{A}))$ and $\text{Mor}(\mathcal{D}(\mathcal{A}))$ is generated by $\text{Mor}(C(\mathcal{A}))$ and formal symbols $x_s : B^\bullet \rightarrow A^\bullet$ for each quasi-isomorphism $s : A^\bullet \rightarrow B^\bullet$, with $x_s \circ s = s \circ x_s = \text{id}$ and $x_{s \circ t} = x_t \circ x_s$. This construction is terrible, because the morphisms in $\mathcal{D}(\mathcal{A})$ are just words like $fx_s g x_t h x_r$, which are unusable. To fix this, we restrict the class of morphisms that we invert to a *localizing class*.

44.3. DEFINITION. A class S of morphisms in a category is called a **localizing class** if

- 1) $s, t \in S \implies st \in S$
- 2) Pullbacks and pushouts exist: Whenever we have a diagram

$$\begin{array}{ccc} & & Z \\ & & \downarrow s \in S \\ X & \xrightarrow{f} & Y \end{array}$$

there exists an object T , a morphism $t \in S$, and a morphism g such that the following diagram commutes:

$$\begin{array}{ccc} T & \xrightarrow{g} & Z \\ \downarrow S \ni t & & \downarrow s \in S \\ X & \xrightarrow{f} & Y \end{array}$$

The similar statement with arrows reversed also holds:

$$\begin{array}{ccc} X & \xrightarrow{s \in S} & Y \\ f \downarrow & & \downarrow f \\ Z & & Z \end{array} \implies \begin{array}{ccc} X & \xrightarrow{s} & Y \\ f \downarrow & & \downarrow g \\ Z & \xrightarrow[t \in S]{} & T \end{array}$$

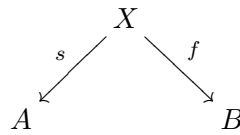
- 3) $fs = gs \implies f = g$ and $sf = sg \implies f = g$, for all $s \in S$.

If S is a localizing class in a category \mathcal{C} , then consider the category \mathcal{D} where $\text{Ob}(\mathcal{D}) = \text{Ob}(\mathcal{C})$ and $\text{Mor}(\mathcal{D})$ is generated by $\text{Mor}(\mathcal{C})$ and all formal inverses of elements of S . Then

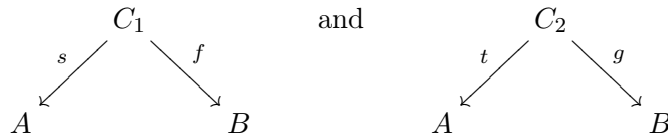
$$\text{Mor}(\mathcal{D}) = \{fx_s \mid s \in S, f \in \text{Mor}(\mathcal{C})\} / \text{some equivalence defined below}$$

The category \mathcal{D} is called the **localization of \mathcal{C} with respect to S** . One key fact about \mathcal{D} is that it is additive.

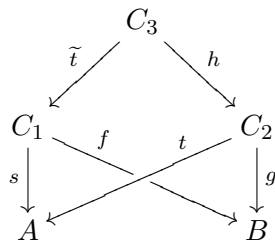
Since x_s is a formal inverse of s , we can represent the morphism $fx_s \in \text{Mor}(\mathcal{D})$ by the diagram



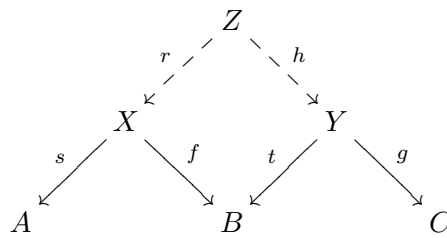
(so $fx_s : A \rightarrow B$). If $fx_s : A \rightarrow B$ and $gx_t : A \rightarrow B$, then we have diagrams



We say that fx_s and gx_t are **equivalent** if there exists a C_3 and a morphism $\tilde{t} \in S$ such that the following diagram commutes

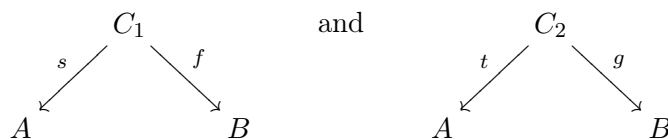


Now suppose we have that $fx_s : A \rightarrow B$ and $gx_t : B \rightarrow C$. The composition $gx_t \circ fx_s$ is defined via the diagram



where the top square exists by item 2) in the definition of localizing class. That is, $gx_t \circ fx_s = gh(x_r x_s)$.

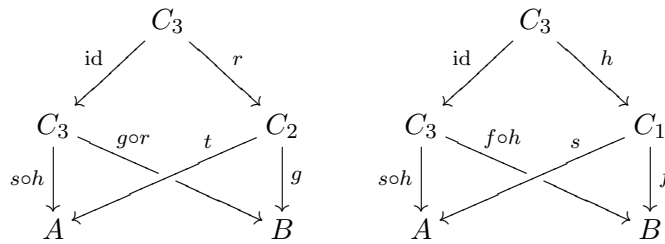
For $fx_s : A \rightarrow B$ and $gx_t : A \rightarrow B$, their sum $fx_s + gx_t$ is defined as follows. Suppose fx_s and gx_t are represented by



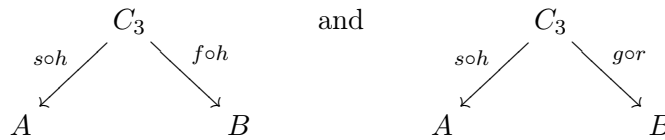
Then from property 2) of localizing classes, there is a diagram

$$\begin{array}{ccc}
 C_3 & \xrightarrow{h} & C_1 \\
 \downarrow S \ni r & & \downarrow s \\
 C_2 & \xrightarrow{t} & A
 \end{array}$$

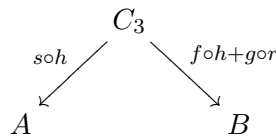
Since this diagram commutes and $t, r \in s$, we get $s \circ h \in S$. We have the following commutative diagrams



Which show that both fx_s and gx_t are equivalent to



respectively. We define $fx_s + gx_t$ to be the morphism $A \rightarrow B$ represented by



Back to constructing the derived category $\mathcal{D}(\mathcal{A})$ of the category of complexes $\mathcal{C}(\mathcal{A})$ of an abelian category \mathcal{A} . We would like to localize $\mathcal{C}(\mathcal{A})$ at the quasi-isomorphisms. The problem is that *quasi-isomorphisms do not form a localizing class in $\mathcal{C}(\mathcal{A})$* . So instead we look at an intermediate category $\mathcal{K}(\mathcal{A})$, where the objects of $\mathcal{K}(\mathcal{A})$ are the same as $\mathcal{C}(\mathcal{A})$, but we identify all morphisms which are homotopic. In $\mathcal{K}(\mathcal{A})$, the quasi-isomorphisms do form a localizing class.

§45. DERIVED CATEGORIES – II. APRIL 20. GUEST LECTURER RINA ANNO. NOTES TYPED BY JULIE.

Let \mathcal{A} be an abelian category, $\mathcal{C}(\mathcal{A})$ the category of complexes of \mathcal{A} , and $\mathcal{D}(\mathcal{A})$ the derived category of \mathcal{A} . Recall that $\mathcal{D}(\mathcal{A})$ is the localization of $\mathcal{K}(\mathcal{A})$ with respect to quasi-isomorphisms, where $\mathcal{K}(\mathcal{A})$ is the category of complexes of \mathcal{A} with morphisms up to homotopy. The objects in each of these categories are the same, so we have maps

$$\mathcal{C}(\mathcal{A}) \rightarrow \mathcal{K}(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{A}).$$

The functor $\mathcal{C}(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{A})$ factors through $\mathcal{K}(\mathcal{A})$. That is, if a morphism $f : A^\bullet \rightarrow B^\bullet$ is homotopic to zero, then it maps to zero in $\mathcal{D}(\mathcal{A})$. The

morphism f factors through an acyclic complex, which is isomorphic to 0 in $\mathcal{D}(\mathcal{A})$. In fact f factors through two acyclic complexes:

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & A^{i-1} & \longrightarrow & A^i & \longrightarrow & A^{i+1} \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \cdots & \longrightarrow & A^{i-1} \oplus A^i & \longrightarrow & A^i \oplus A^{i+1} & \longrightarrow & A^{i+1} \oplus A^{i+2} \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \cdots & \longrightarrow & B^{i-2} \oplus B^{i-1} & \longrightarrow & B^{i-1} \oplus B^i & \longrightarrow & B^i \oplus B^{i+1} \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \cdots & \longrightarrow & B^{i-1} & \longrightarrow & B^i & \longrightarrow & B^{i+1} \longrightarrow \cdots
 \end{array}$$

where the maps in the second and third row are the identity on one component and zero on the other. We leave it to the reader to construct the downward maps so that all squares are commutative and so that the composition is f .

We have seen that morphisms in $\mathcal{D}(\mathcal{A})$ have the form

$$\begin{array}{ccc}
 & \tilde{A}^\bullet & \\
 \text{qis} \swarrow & & \searrow f \\
 A^\bullet & & B^\bullet
 \end{array}$$

where f is defined up to homotopy. This means that commutative diagrams in $\mathcal{D}(\mathcal{A})$ lift to diagrams in $\mathcal{C}(\mathcal{A})$ which only commute *up to homotopy*. This obstacle makes $\mathcal{D}(\mathcal{A})$ seemingly not ideal to work with. Why does it make sense to consider complexes up to quasi-isomorphisms? The answer is resolutions.

Given $A \in \mathcal{C}(\mathcal{A})$, we can think of it as the complex

$$0 \longrightarrow 0 \longrightarrow A \longrightarrow 0 \longrightarrow \cdots$$

in $\mathcal{D}(\mathcal{A})$. This gives an equivalence of \mathcal{A} with a subcategory of $\mathcal{D}(\mathcal{A})$. Morphisms $A \rightarrow B$ in $\mathcal{C}(\mathcal{A})$ are the same as morphisms

$$\begin{array}{ccccccc}
 0 & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 \longrightarrow \cdots \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & 0 & \longrightarrow & B & \longrightarrow & 0 \longrightarrow \cdots
 \end{array}$$

in $\mathcal{D}(\mathcal{A})$.

Take a resolution of A in \mathcal{A}

$$\cdots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow A \longrightarrow 0 \longrightarrow 0$$

Truncation gives us a map to A thought of as an element of $\mathcal{D}(\mathcal{A})$:

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & P^{-2} & \longrightarrow & P^{-1} & \longrightarrow & P^0 \longrightarrow 0 \longrightarrow 0 \\
 & & & & & & \downarrow \text{qis} \\
 0 & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 \longrightarrow \cdots
 \end{array}$$

This map is a quasi-isomorphism, and so $A \in \mathcal{A}$ is isomorphic in $\mathcal{D}(\mathcal{A})$ to its resolutions. One of the benefits of this is that it's now easy to define derived homomorphisms $A \rightarrow B$.

Let us discuss some properties of $\mathcal{D}(\mathcal{A})$.

45.1. PROPERTY. $\mathcal{D}(\mathcal{A})$ is additive.

45.2. PROPERTY. There exists a subcategory in $\mathcal{D}(\mathcal{A})$ equivalent to \mathcal{A} , namely the category of complexes with only one nonzero term.

45.3. PROPERTY. There exists a translation functor $T : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{A})$ which takes A^\bullet to $A^\bullet[1]$.

The translation functor comes from the functor $T : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A})$, where $T(A^\bullet)^i = A^{i+1}$. This functor filters through the map $\mathcal{C}(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{A})$.

Recall the definition of $R^i \text{Hom}(A, B)$ for A, B objects of \mathcal{A} :

(1) Take an injective resolution of B :

$$B \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

(2) Take $\text{Hom}_{\mathcal{A}}(A, \bullet)$ of the truncated sequence:

$$\text{Hom}(A, I^0) \longrightarrow \text{Hom}(A, I^1) \longrightarrow \text{Hom}(A, I^2) \longrightarrow \cdots$$

(3) $R^i \text{Hom}(A, B)$ is the i^{th} cohomology of this complex of abelian groups.

In $\mathcal{D}(\mathcal{A})$, B is isomorphic to its injective resolution $0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$. Consider

$$\text{Hom}_{\mathcal{D}(\mathcal{A})}(A, I^\bullet),$$

where by abuse of notation we denote $0 \longrightarrow A \longrightarrow 0 \longrightarrow \cdots$ by A . These are morphisms

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots \end{array}$$

up to homotopy. But since there is no homotopy, they are simply maps of complexes. We see that

$$\text{Hom}_{\mathcal{D}(\mathcal{A})}(A, I^\bullet) = \{A \rightarrow I^0 \mid A \rightarrow I^0 \rightarrow I^1 \text{ is zero} \}$$

Next consider

$$\text{Hom}_{\mathcal{D}(\mathcal{A})}(A, I^\bullet[1]).$$

These are morphisms (up to homotopy)

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots \end{array}$$

Such a morphism is homotopy equivalent to zero if we have a morphism $A \rightarrow I^0$ such that the composition $A \rightarrow I^0 \rightarrow I^1$ is zero. Thus

$$\text{Hom}_{\mathcal{D}(\mathcal{A})}(A, I^\bullet[1]) = \{A \rightarrow I^1 \mid A \rightarrow I^1 \rightarrow I^2 = 0\} / \{\text{compositions } A \rightarrow I^0 \rightarrow I^1\}.$$

In general, we have

$$\text{Hom}_{\mathcal{D}(\mathcal{A})}(A, I^\bullet[n]) = \{A \rightarrow I^n \mid A \rightarrow I^n \rightarrow I^{n+1} = 0\} / \{\text{compositions } A \rightarrow I^{n-1} \rightarrow I^n\}.$$

Since B is isomorphic in $\mathcal{D}(\mathcal{A})$ to $I[n]$, we have

45.4. PROPERTY. $\text{Hom}_{\mathcal{D}(\mathcal{A})}(A, B[n]) = R^n \text{Hom}_{\mathcal{A}}(A, B) = \text{Ext}^n(A, B)$ for A and B objects of \mathcal{A} .

We still have a couple of issues with $\mathcal{D}(\mathcal{A})$ that we want to address:

- We would like a subcategory of $\mathcal{K}(\mathcal{A})$ such that morphism between the corresponding complexes in $\mathcal{D}(\mathcal{A})$ are morphisms of complexes.
- We would like to extend the notion of an injective resolution to all of $\mathcal{D}(\mathcal{A})$.

Is it true that any object in $\mathcal{D}(\mathcal{A})$ is quasi-isomorphic to a complex of injective objects? Recall that if \mathcal{A} has enough injective objects, then we can construct injective resolutions term-by-term. Let $\mathcal{D}^+(\mathcal{A})$ be the set of complexes in $\mathcal{D}(\mathcal{A})$ which are bounded from the right, $\mathcal{D}^-(\mathcal{A})$ those which are bounded from the left, and $\mathcal{D}^b(\mathcal{A})$ complexes which are bounded from both sides.

45.5. PROPOSITION. *Given an object K^\bullet of $\mathcal{D}^+(\mathcal{A})$, there exists a morphism $K^\bullet \rightarrow I^\bullet$ of complexes where I^\bullet is bounded from the left and consists of injective objects.*

Idea of proof. Since \mathcal{A} has enough injective objects, we have a morphism $K^0 \rightarrow I^0$ where I^0 is injective. To construct I^1 , use the fact that pushouts exist in $\mathcal{D}(\mathcal{A})$ to find an object Z so that the diagram

$$\begin{array}{ccc} K^0 & \longrightarrow & K^1 \\ \downarrow & & \downarrow \\ I^0 & \dashrightarrow & Z \end{array}$$

is commutative. Embedding Z into an injective object I^1 gives the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K^0 & \longrightarrow & K^1 & \longrightarrow & K^2 \longrightarrow \dots \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & & \end{array}$$

Repeating this process gives the desired morphism $K^\bullet \rightarrow I^\bullet$. □

Notice that this proof relies heavily on the fact that K^\bullet is bounded from the left. For unbounded complexes, a similar result involving the existence of a “ K -injective resolution” was proved by Spaltenstein (‘88).

All classical functors, for instance $A \otimes \bullet$, $\text{Hom}(A, \bullet)$, f_* , f^* , $f_!$, etc. can be extended to $\mathcal{D}^b(\mathcal{A})$ by replacing objects by injective or flat resolutions.

45.6. PROPERTY. *Derived functors can be put together to form functors between derived categories.*

For example, consider $f_* : \mathcal{A} \rightarrow \mathcal{B}$ and construct $R^i f_* : \mathcal{A} \rightarrow \mathcal{B}$. Because of the way $R^i f_*$ is constructed, when we extend to derived categories we get the i^{th} cohomology of $R^i f_*$.

In abelian categories we have short exact sequences. Derived categories are not abelian, but they have a structure arising from short exact sequences of complexes. Given $f : K^\bullet \rightarrow L^\bullet$, we can construct a complex $C(f)$ called the *cone of f* , where $C(f)^i = K^{i+1} \oplus L^i$ with differential d as in the following diagram:

$$\begin{array}{ccc} K^{i+1} & \xrightarrow{\oplus} & L^i \\ d_K \downarrow & \searrow f & \downarrow d_L \\ K^{i+2} & \xrightarrow{\oplus} & L^{i+1} \end{array}$$

There exists a canonical map $C(f) \rightarrow K[1]$ coming from the map $L \rightarrow C(f)$.

45.7. DEFINITION. A diagram $A \rightarrow B \rightarrow C \rightarrow A[1]$ in $\mathcal{D}(\mathcal{A})$ or $\mathcal{K}(\mathcal{A})$ is a *distinguished triangle* if it is quasi-isomorphic to a triangle of the form

$$A \xrightarrow{f} B \longrightarrow C(f) \longrightarrow A[1]$$

Distinguished triangles have properties similar to short exact triples. For instance, they produce a long exact sequence of homology. The key difference is that we can form exact triangles for any morphism $A \rightarrow B$. They give us a way to take quotients of noninjective morphisms.

45.8. PROPERTY. $\mathcal{D}(\mathcal{A})$ is a *triangulated cone*, i.e. a quotient of a noninjective morphism.

§46. HOME STRETCH: CAP PRODUCT, POINCARÉ DUALITY, SERRE DUALITY.

HOMEWORK 1. DEADLINE: FEBRUARY 10.

Sign the worksheet and bring it to my office hours when you present solutions orally. Staple it to your written solutions when you turn them in. There will be six homework sets and you have to accumulate 90 points to get an A in this class. All rings are associative and contain 1.

Problem 1. In the exact sequences of complexes

$$0 \rightarrow X^\bullet \rightarrow Y^\bullet \rightarrow Z^\bullet \rightarrow 0,$$

show that if X^\bullet and Z^\bullet are acyclic then Y^\bullet is acyclic (1 point).

Problem 2. Consider a commutative diagram of R -modules

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 & \longrightarrow & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & Z_1 & \longrightarrow & Z_2 & \longrightarrow & Z_3 & \longrightarrow & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & 0 & & & & \end{array}$$

Suppose the columns and the two bottom rows are exact. Show that the top row is exact (1 point).

Problem 3. Consider R -linear maps of R -modules.

$$X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3. \tag{46.1}$$

Suppose the induced sequence

$$\text{Hom}_R(X_3, Y) \rightarrow \text{Hom}_R(X_2, Y) \rightarrow \text{Hom}_R(X_1, Y)$$

is exact for any R -module Y . Show that (46.1) is also exact (1 point).

Problem 4. (Five Lemma). Given a commutative diagram of R -linear maps with exact rows

$$\begin{array}{ccccccccc} X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & X_4 & \longrightarrow & X_5 \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\ Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 & \longrightarrow & Y_4 & \longrightarrow & Y_5 \end{array}$$

suppose f_1 is an epimorphism, f_5 is a monomorphism, f_2 and f_4 are isomorphisms. Show that f_3 is an isomorphism (1 point).

Problem 5. Show that an R -module P is projective \Leftrightarrow any exact sequence $0 \rightarrow M' \rightarrow M'' \rightarrow P \rightarrow 0$ splits (i.e. $M'' \simeq M' \oplus P$) \Leftrightarrow the functor $M \mapsto \text{Hom}_R(P, M)$ is exact (i.e. takes short exact sequences to short exact sequences) (1 points).

Problem 6. Let R be a PID and let M be a finitely generated R -module. Show that M is projective $\Leftrightarrow M$ is free $\Leftrightarrow M$ is torsion-free (1 point).

Problem 7. Let M be a projective (resp. injective) R -module, where R is commutative. Let $S \subset R$ be a multiplicative system. Show that $S^{-1}M$ is a projective (resp. injective) $S^{-1}R$ -module (1 point).

Problem 8. Let R be a PID. Show that an R -module M is injective if and only if it is divisible (1 point).

Problem 9. Consider the following complex

$$C = [\dots \rightarrow 0 \rightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \rightarrow 0 \rightarrow 0 \rightarrow \dots]$$

Show that C is not a projective object in the category of complexes of Abelian groups, even though each of its terms is a projective Abelian group (1 point).

Problem 10. (a) Define a category Δ with objects given by natural numbers $\mathbf{Ob}(\Delta) = \{0, 1, 2, \dots\}$. The set of morphisms $\mathbf{Mor}(m, n)$ is the set of all non-decreasing maps from $\{0, 1, \dots, m\}$ to $\{0, 1, \dots, n\}$. (b) A contravariant functor $X : \Delta \rightarrow \mathbf{Sets}$ is called a *simplicial set*. Describe what a simplicial set is without the categorical language. (c) Let Y be a topological space. Show that we can define a simplicial set as follows. For any $n \in \{0, 1, 2, \dots\} = \mathbf{Ob}(\Delta)$, let X_n be the set of all singular simplices, i.e. the set of all continuous maps $\Delta_n \rightarrow Y$, where Δ_n is the standard n -dimensional simplex. Come up with a reasonable function $X_n \rightarrow X_m$ for each non-decreasing map from $\{0, 1, \dots, m\}$ to $\{0, 1, \dots, n\}$. (d) For any simplicial set X and Abelian group A , define homology groups $H_i(X, A)$ which agree with singular homology of the topological space Y if X is defined in (c). (e) For any simplicial complex Y , define a simplicial set X such that its homology defined in (d) agrees with simplicial homology of a simplicial complex (2 points).

Problem 11. Triangulate and compute simplicial homology (with coefficients in \mathbb{Q}) of the following spaces (2 points):

- (a) two-dimensional torus;
- (b) Möbius band;
- (c) Klein bottle;
- (d) real projective plane.

For a bonus point, compute homology with integer coefficients.

Problem 12. Triangulate and compute simplicial homology (with coefficients in \mathbb{Q}) of the following spaces (2 points):

- (a) n -dimensional simplex;
- (b) n -dimensional sphere.

Problem 13. (Mayer–Vietoris) Let M be a differentiable manifold covered by two open sets $M = U \cup V$. Consider the following sequence

$$0 \rightarrow \Omega_{dR}^k(M) \xrightarrow{(i_U^*, i_V^*)} \Omega_{dR}^k(U) \oplus \Omega_{dR}^k(V) \xrightarrow{i_1^* - i_2^*} \Omega_{dR}^k(U \cap V) \rightarrow 0,$$

where $i_U : U \rightarrow M$, $i_V : V \rightarrow M$, $i_1 : U \cap V \rightarrow U$, $i_2 : U \cap V \rightarrow V$ are inclusions. (a) Show that this sequence is exact. (Hint: use partition of unity.) (b) Write down the corresponding long exact sequence of cohomology. It is called the *Mayer–Vietoris sequence*. (c) Use the Mayer–Vietoris

sequence and homotopy invariance of de Rham cohomology to compute de Rham cohomology of the sphere S^n (2 points).

Problem 14. For each of the following mappings, decide if it is an Euler–Poincare mapping. (a) In \mathbf{Ab} , let $\phi(A)$ be the rank of A if A is finitely generated, undefined otherwise. (b) Let R be a domain with a quotient field K . In $R\text{-mod}$, let $\phi(A) = \dim_K(A \otimes_R K)$ if the latter vector space is finite-dimensional, undefined otherwise. (c) In $\mathbf{Kom}(\mathbf{Vect}_k)$, let $\phi(A) = \sum_{n \in \mathbb{Z}} (\dim_k A_n) q^n$ if all A_n are finite-dimensional, undefined otherwise. Here ϕ takes values in formal Laurent series $\mathbb{Z}[[q^{-1}, q]]$ (2 points).

Problem 15. Let M be an R -module, where R is commutative. Suppose that for every ideal $I \subset R$, any R -linear map $I \rightarrow M$ can be extended to an R -linear map $R \rightarrow M$. Show that M is injective (2 points).

Problem 16. Let R be a Dedekind domain and let M be a finitely generated R -module. Show that M is projective $\Leftrightarrow M$ is torsion-free (3 points).

Problem 17. Let \mathfrak{g} be a Lie algebra (over a field k). Consider a sequence

$$\mathfrak{g}^* \xrightarrow{d} \Lambda^2 \mathfrak{g}^* \xrightarrow{d} \Lambda^3 \mathfrak{g}^* \xrightarrow{d} \dots,$$

where $\Lambda^r \mathfrak{g}^*$ is the space of skew-symmetric r -linear forms on \mathfrak{g} and the differential is defined as follows:

$$dc(g_1, \dots, g_{r+1}) = \sum_{1 \leq i < j \leq r+1} (-1)^{i+j-1} c([g_i, g_j], g_1, \dots, \hat{g}_i, \dots, \hat{g}_j, \dots, g_{r+1}).$$

(a) Show that d is well-defined (i.e. takes skew-symmetric forms to skew-symmetric forms). (b) Show that $d^2 = 0$. (c) Cohomology groups $H^i(\mathfrak{g}, k)$ of this complex are called cohomology groups of a Lie algebra (with trivial coefficients). Compute $H^\bullet(\mathfrak{sl}_2, k)$ (3 points).

Problem 18. (Bar complex - do this exercise if you are interested in future exercises on Hochschild cohomology). Let k be a commutative ring and let A be a k -algebra. Consider the following sequence of k -modules and k -linear maps

$$\dots A^{\otimes 4} \xrightarrow{b_3} A^{\otimes 3} \xrightarrow{b_2} A^{\otimes 2} \xrightarrow{b_1} A, \tag{46.2}$$

where all tensor products are over k . Here b_1 is the multiplication map and, more generally, the maps $b_n : A^{\otimes(n+1)} \rightarrow A^{\otimes n}$ are defined as follows:

$$b_n(a_0 \otimes a_1 \otimes \dots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \dots \otimes (a_i a_{i+1}) \otimes \dots \otimes a_n.$$

(a) Show that (46.2) is a complex. (b) Show that (46.2) is an exact sequence by checking that the map

$$h(a_1 \otimes \dots \otimes a_n) = 1 \otimes a_1 \otimes \dots \otimes a_n$$

provides a homotopy, i.e. $bh + hb = \text{Id}$ (3 points).

HOMEWORK 2. DEADLINE: FEBRUARY 24.

Problem 1. Find the left adjoint functors of the following forgetful functors (a) $\mathbf{Ab} \rightarrow \mathbf{Sets}$; (b) $\mathbf{Top} \rightarrow \mathbf{Sets}$; (c) (Associative k -algebras) $\rightarrow \mathbf{Vect}_k$ (1 point).

Problem 2. Find the left adjoint functors of the following functors that give fully faithful embeddings of categories. Part of this problem is to define these categories (especially morphisms!) (a) $\mathbf{Fields} \rightarrow \mathbf{Domains}$; (b) $\mathbf{Ab} \rightarrow \mathbf{Groups}$; (c) $\mathbf{CompleteMetricSpaces} \rightarrow \mathbf{MetricSpaces}$ (1 point).

Problem 3. Find a right adjoint functor of (a) A functor $\mathbf{Top} \rightarrow \mathbf{Top}$ that maps X to $X \times S^1$; (b) duality functor $*$: $\mathbf{Vect}_k \rightarrow (\mathbf{Vect}_k)^{op}$ (1 point).

Problem 4. Give an example of a functor that has neither left nor right adjoint. (1 point).

Problem 5. Let M be an R -module. Show that any two injective resolutions of M are homotopy equivalent. (1 point)

Problem 6. Show that any projective R -module is flat. Is any injective R -module flat? (1 point)

Problem 7. Let $Z \subset Y \subset X$ be closed subspaces of a topological space. Show that there exists an exact sequence of relative singular homology groups

$$\dots \rightarrow H_i(X, Y; \mathbb{Z}) \rightarrow H_{i-1}(Y, Z; \mathbb{Z}) \rightarrow H_{i-1}(X, Z; \mathbb{Z}) \rightarrow H_{i-1}(X, Y; \mathbb{Z}) \rightarrow \dots$$

(1 point).

Problem 8. Let G be a divisible Abelian group. Show that $G = V \oplus T$, where V is a \mathbb{Q} -vector space and T is a torsion subgroup of G (1 point).

Problem 9. Let $H \subset G$ be finite groups. (a) Consider the following functors between categories of finite-dimensional representations:

$$\mathbf{Res} : G\text{-mod} \rightarrow H\text{-mod} \quad \text{and} \quad \mathbf{Ind} : H\text{-mod} \rightarrow G\text{-mod},$$

where \mathbf{Res} is a forgetful functor and \mathbf{Ind} sends an H -module V to an induced G -module $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$. Show that $\mathbf{Ind} \dashv \mathbf{Res}$ (Frobenius reciprocity).

(b) Let W be an irreducible G -module. Show that $\dim W^H$ is equal to the multiplicity of W in $\mathbb{C}[G/H]$ (formal linear combinations of cosets with a natural G -action). (1 point).

Problem 10. Consider a functor $F : \mathcal{A} \rightarrow \mathcal{B}$ between Abelian categories. Show that if F has a right adjoint then F is right exact, and if F has a left adjoint then F is left exact. (2 points).

Problem 11. Let R be a commutative ring. For any R -module E , let $E^\vee = \text{Hom}_{\mathbb{Z}}(E, \mathbb{Q}/\mathbb{Z})$ be the dual R -module. (a) Show that $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is exact if and only if $0 \rightarrow C^\vee \rightarrow B^\vee \rightarrow A^\vee \rightarrow 0$ is exact. (b) Let F be a flat R -module and let I be an injective R -module. Show that $\text{Hom}_R(F, I)$ is injective. (c) Show that an R -module E is flat if and only if E^\vee is injective. (2 points)

Problem 12. (a) Let X be a topological space. We define a cone CX over X as a topological space obtained from the cylinder $X \times [0, 1]$ by identifying all points in $X \times \{1\}$. The topology is the weakest topology such that the

map $X \times [0, 1] \rightarrow CX$ is continuous. Show that $H_0^{sing}(CX, \mathbb{Z}) = \mathbb{Z}$ and $H_i^{sing}(CX, \mathbb{Z}) = 0$ for $i > 0$ (b) Let X be a simplicial complex. We define a cone CX over X as a simplicial complex with one extra vertex $*$ (the apex of the cone). The n -simplices of CX are either n -simplices of X or subsets $I \cup \{*\}$, where I is an $(n - 1)$ -simplex of X . Show that $H_0^{simp}(CX, \mathbb{Z}) = \mathbb{Z}$ and $H_i^{simp}(CX, \mathbb{Z}) = 0$ for $i > 0$. (2 points).

Problem 13. Consider short exact sequences $0 \rightarrow N \rightarrow P \rightarrow M \rightarrow 0$ and $0 \rightarrow N' \rightarrow P' \rightarrow M \rightarrow 0$, with P and P' projective. Show that $P \oplus N' \simeq P' \oplus N$ (2 points).

Problem 14. This problem relies heavily on Problem 10 from HW 1. Let G be a group. For any $n \geq 0$, let $(BG)_n = G^n$. For any non-decreasing monotonous map $f : \{0, 1, \dots, m\} \rightarrow \{0, 1, \dots, n\}$, let

$$BG(f) : (BG)_n \rightarrow (BG)_m, \quad BG(f)(g_1, \dots, g_n) = (h_1, \dots, h_m),$$

where

$$h_i = \prod_{j=f(i-1)+1}^{f(i)} g_j, \quad h_i = e \quad \text{if} \quad f(i-1) = f(i).$$

(a) Show that BG is a simplicial set. Its geometric realization is called the *classifying space* of G . (b) Now also fix an Abelian group A . Write down an explicit complex that computes $H_\bullet(BG, A)$. (2 points).

Problem 15. Let A be an Abelian category. Its K -group $K(A)$ is an Abelian group generated by symbols $[M]$ for each object M of A , subject to relations $[M] = [M'] + [M'']$ for each exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$. Compute K -groups of the category of (a) finite-dimensional k -vector spaces; (b) finitely generated modules over $\mathbb{C}[x]$; (c) finite-dimensional complex representations of S_3 . (2 points).

Problem 16. Consider functors $F : \mathcal{C} \rightarrow \mathcal{D}$, $G : \mathcal{D} \rightarrow \mathcal{C}$ between categories. Show that they are adjoint if and only if there exist natural transformations

$$\sigma : FG \rightarrow \text{Id}_{\mathcal{D}}, \quad \tau : \text{Id}_{\mathcal{C}} \rightarrow GF$$

such that compositions

$$F \xrightarrow{F \circ \tau} FGF \xrightarrow{\sigma \circ F} F \quad \text{and} \quad G \xrightarrow{\tau \circ G} GFG \xrightarrow{G \circ \sigma} G$$

are identity transformations (3 points).

Problem 17. Let \mathbf{Kom}^+ be the category of cochain complexes (C^\bullet, d) of R -modules such that $C^i = 0$ for $i < 0$. Show that (C^\bullet, d) is a projective object of \mathbf{Kom}^+ if and only if the following conditions are satisfied: (1) every C^n is projective; (2) $H^n(C^\bullet) = 0$ for $n > 0$; (3) $\text{Ker } d_n$ is a direct summand of C_n for any n . (3 points).

Problem 18. Show that the category \mathbf{Kom}^+ of the previous exercise has enough projective objects (3 points).

HOMEWORK 3. DEADLINE: MARCH 9.

In these exercises R is a commutative ring with unity and M is an R -module.

Problem 1. Consider cohomology $H^\bullet(C)$ of the cochain complex C^\bullet as a complex with trivial differentials. Is it true that C^\bullet and $H^\bullet(C)$ are always homotopy equivalent? (1 point).

Problem 2. Suppose R is Noetherian and let $K(x_1, \dots, x_n)$ be the Koszul complex of elements $x_1, \dots, x_n \in R$. (a) Let $r(x_1, \dots, x_n)$ be the minimal integer r such that $H^r K(x_1, \dots, x_n) \neq 0$. Show that

$$r(x_1, \dots, x_n) = r(x_1, \dots, x_n, 0, \dots, 0).$$

(b) Suppose x_1, \dots, x_s is a maximal regular sequence in the ideal (x_1, \dots, x_n) . Then

$$H^i K(x_1, \dots, x_n) = \begin{cases} 0 & i < s \\ \frac{(x_1, \dots, x_s) : (x_1, \dots, x_n)}{(x_1, \dots, x_s)} \neq 0 & i = s \end{cases}$$

(c) Suppose that $y_1, \dots, y_s \in (x_1, \dots, x_n)$. Show that Koszul complexes $K(x_1, \dots, x_n, y_1, \dots, y_s)$ and $K(x_1, \dots, x_n, 0, \dots, 0)$ (s zeros) are isomorphic.

(d) Finish the proof of Theorem 18.10 (3 points).

Problem 3. (a) Let $x_1, \dots, x_n \in R$ be a regular sequence. Show that $x_1^{a_1}, \dots, x_n^{a_n}$ is a regular sequence for any integers a_1, \dots, a_n assuming R is local. (b) The same problem but for any R . (2 points).

Problem 4. Let $R = k[x_1, \dots, x_n]$. We can view k as an R -module via $k = R/(x_1, \dots, x_n)$. Compute $\text{Tor}_i^R(k, k)$ (1 point).

Problem 5. Let $I, J \subset R$. Show that $\text{Tor}_0^R(R/I, R/J) \simeq R/(I + J)$, $\text{Tor}_1^R(R/I, R/J) \simeq (I \cap J)/(IJ)$ (1 point).

Problem 6. Compute $\text{Tor}_i^{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2)$ (1 point).

Problem 7. Compute $\text{Tor}_i^{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}_m)$, $\text{Ext}_{\mathbb{Z}}^i(\mathbb{Z}_n, \mathbb{Z}_m)$ (1 point).

Problem 8. Show that the following conditions are equivalent (2 points)

- M is projective.
- $\text{Ext}^i(M, N) = 0$ for $i > 0$ and any R -module N .
- $\text{Ext}^1(M, N) = 0$ for any R -module N .

Problem 9. Show that the following conditions are equivalent (1 point)

- M is flat.
- $\text{Tor}_i(N, M) = 0$ for any $i > 0$ and any R -module N .

Problem 10. Show that the following conditions are equivalent (2 points)

- M is flat.
- $I \otimes M \rightarrow IM$, $a \otimes m \mapsto am$, is an isomorphism for any ideal $I \subset R$.
- $\text{Tor}_1(R/I, M) = 0$ for any ideal $I \subset R$.
- $\text{Tor}_1(N, M) = 0$ for any finitely generated R -module N .

Problem 11. (2 points) Let $I, J \subset R = k[x_1, x_2, x_3, x_4]$ be the following ideals:

$$I = (x_1 - x_3, x_2 - x_4), \quad J = (x_1, x_2) \cap (x_3, x_4).$$

Compute

$$\sum_{i \geq 0} (-1)^i \dim_k \text{Tor}_i^R(R/I, R/J).$$

(Remark: in algebraic geometry, this expression (*Serre's multiplicity formula*) computes the intersection number of subvarieties $V(I), V(J) \subset \mathbb{A}^4$.)

Problem 12. Let B be an Abelian group and let $T(B) \subset B$ be the torsion subgroup (consisting of all elements of finite order). Show that $T(B) \simeq \text{Tor}_1^{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})$ (1 point).

In exercises 13 and 14, we consider a mapping cone $M(\alpha)$ of a map of chain complexes $\alpha : F_{\bullet} \rightarrow G_{\bullet}$ (as opposed to cochain complexes considered in class). It is defined as $M(\alpha)_i = F_{i-1} \oplus G_i$, with a differential given by the matrix $\begin{bmatrix} -d_F & 0 \\ -\alpha & d_G \end{bmatrix}$.

Problem 13. By analogy with a topological mapping cone described in class, define a mapping cone Z for an inclusion $f : X \rightarrow Y$ of simplicial complexes. Z should be a simplicial complex. Let M_{\bullet} be a mapping cone of a map of chain complexes $C_{\bullet}^{\text{simp}}(X, \mathbb{Z}) \rightarrow C_{\bullet}^{\text{simp}}(Y, \mathbb{Z})$. Show that we have an exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow C_{\bullet}^{\text{simp}}(X, \mathbb{Z}) \rightarrow M_{\bullet} \rightarrow 0$$

where \mathbb{Z} is a complex concentrated in degree 0 (1 point).

Problem 14. Given a map $f : B \rightarrow C$ of chain complexes, let v denote the inclusion of C into $M(f)$. Show that there is a chain homotopy equivalence $M(v) \rightarrow B[-1]$ (1 point).

Problem 15. Let R be a k -algebra with 1 over a field k . Recall that M is called an R -bimodule if it has structures of both a left and a right R -module such that $(rm)s = r(ms)$ for any $r, s \in R, m \in M$. An example of an R -bimodule is R itself. (a) Show that an R -bimodule M is the same thing as a R^e -module, where $R^e := R \otimes_k R^{\text{op}}$ is called an *enveloping algebra* of R . Here R^{op} is equal to R as a set, but has an opposite multiplication $r \circ s = sr$. We define *Hochschild cohomology*

$$HH^n(R, M) = \text{Ext}_{R^e}^n(R, M).$$

(b) To compute Hochschild cohomology, we need a projective resolution of R as an R -bimodule (i.e. as a R^e -module). Show that one choice is given by the bar resolution (46.2). (c) More concretely, deduce that $HH^n(R, M)$ can be computed as cohomology of a complex $C^n = \text{Hom}_k(R^{\otimes n}, M)$ (and $C^0 = M$) with the differential $d : C^n \rightarrow C^{n+1}$ given by formula

$$\begin{aligned} (df)(a_1 \otimes \dots \otimes a_{n+1}) &= a_1 f(a_2 \otimes \dots \otimes a_{n+1}) \\ &\quad - f((a_1 a_2) \otimes \dots \otimes a_{n+1}) + f(a_1 \otimes (a_2 a_3) \otimes \dots \otimes a_{n+1}) \\ &\quad + \dots + (-1)^n f(a_1 \otimes \dots \otimes (a_n a_{n+1})) - (-1)^n f(a_1 \otimes \dots \otimes a_n) a_{n+1}. \end{aligned}$$

(2 points)

Problem 16. [Hochschild – continued] (a) Show that

$$HH^0(R, M) = \{m \in M \mid rm = mr\}.$$

(b) A map $f : R \rightarrow M$ is called a *derivation* if it satisfies the Leibniz rule $f(rs) = rf(s) + f(r)s$. An example is an inner derivation, which has form $f(r) = rm - mr$ for some $m \in R$. Show that

$$HH^1(R, M) = \{\text{derivations } R \rightarrow M\} / \{\text{inner derivations}\}.$$

(c) Compute $HH^1(k[x], k[x])$. (2 points)

Problem 17. [Hochschild – continued] Let $k[[q]]$ be the ring of formal power series in q . Let $R[[q]] = R \otimes_k k[[q]]$. A *formal deformation* of R is a unital $k[[q]]$ -algebra structure on $R[[q]]$ such that the product

$$R[[q]] \otimes_{k[[q]]} R[[q]] \rightarrow R[[q]]$$

has the form

$$a \otimes b \mapsto ab + m_1(a, b)q + m_2(a, b)q^2 + \dots$$

for any $a, b \in R$ (here $m_i(a, b) \in R$). Show that the class of m_1 belongs to $HH^2(R, R)$ (and so this Hochschild cohomology group parametrises infinitesimal deformations of R) (1 point)

HOMEWORK 4. DEADLINE: MARCH 30.

Problem 0. Suppose $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ is an additive functor between abelian categories that sends short exact sequences to short exact sequences. Show that \mathcal{F} sends arbitrary exact sequences to exact sequences. (1 point).

Problem 1. Prove the inductive step in Claim 22.7 (1 point).

Problem 2. Prove the base of induction in Claim 22.9 (1 point)

Problem 3. Consider an abelian category with enough projective objects.

A chain complex $\dots \rightarrow Q_2 \xrightarrow{d_2} Q_1 \xrightarrow{d_1} Q_0 \rightarrow 0$ is called *split-exact* if the following is true: each $Q_i \simeq A_i \oplus B_i$, $d_i|_{A_i} = 0$, $d_i|_{B_i}$ is an isomorphism with A_{i-1} , and $B_0 = 0$. Using horseshoe lemma, show that for any exact sequence

$$\dots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0 \rightarrow 0, \tag{46.3}$$

there exist projective resolutions $\dots \rightarrow P_{2i} \rightarrow P_{1i} \rightarrow P_{0i} \rightarrow X_i \rightarrow 0$ of each X_i , and maps of chain complexes $\dots \rightarrow P_{\bullet 2} \rightarrow P_{\bullet 1} \rightarrow P_{\bullet 0} \rightarrow 0$ extending (46.3), such that the sequence $\dots \rightarrow P_{k,2} \rightarrow P_{k,1} \rightarrow P_{k,0} \rightarrow 0$ is split exact for any k (1 point).

Problem 4. Let $\mathcal{F} : R\text{-mod} \rightarrow S\text{-mod}$ be an additive functor. Show that \mathcal{F} commutes with taking direct sums. More generally, show that \mathcal{F} takes split-exact sequences to split-exact sequences (1 point).

Problem 5. Let \mathcal{A} be an abelian category with enough projective objects. Let $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ be a covariant right-exact functor. An object $A \in \mathcal{A}$ is called \mathcal{F} -acyclic if $L_i\mathcal{F}(A) = 0$ for any $i > 0$. (a) Let (46.3) be an exact sequence of \mathcal{F} -acyclic objects. Show that $\dots \rightarrow \mathcal{F}(X_2) \rightarrow \mathcal{F}(X_1) \rightarrow \mathcal{F}(X_0)$ is also exact. (b) Let A be any object of \mathcal{A} and consider its \mathcal{F} -acyclic resolution $\dots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0 \rightarrow A \rightarrow 0$, i.e. an exact sequence where all objects X_i are \mathcal{F} -acyclic. Show that $L_i\mathcal{F}(A)$ is isomorphic to the i -th homology of the complex $\dots \rightarrow \mathcal{F}(X_2) \rightarrow \mathcal{F}(X_1) \rightarrow \mathcal{F}(X_0) \rightarrow 0$. Formulate consequences for computing Tor (2 points).

Problem 6. In this exercise we explore interpretation of $\text{Ext}^1(M, N)$ as isomorphism classes of extensions $0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0$. We work in the category $R\text{-mod}$. (a) Construct operations on extensions that reflect the fact that $\text{Ext}^1(M, N)$ is a contravariant functor of M and a covariant functor of N . (b) Given two extensions E and E' of N by M , we can take direct sums to obtain $0 \rightarrow N \oplus N \rightarrow E \oplus E' \rightarrow M \oplus M \rightarrow 0$. Using functoriality with respect to the diagonal map $N \rightarrow N \oplus N$ and the “summation” map $M \oplus M \rightarrow M$, $(m_1, m_2) \mapsto m$, construct an extension of N by M , denoted $E \oplus E'$ (called the *Baer sum*). Show that it corresponds to the additive structure on $\text{Ext}^1(M, N)$ (2 points).

Problem 7. Consider the “horizontal” spectral sequence of a double complex. Show that an element $x_0 \in K^{p,q}$ represents an element of $E_r^{p,q}$ if and only if there exists a *zig-zag*

$$x_1 \in K^{p+1,q-1}, \quad x_2 \in K^{p+2,q-2}, \dots, \quad x_{r-1} \in K^{p+r-1,q-r+1}$$

such that

$$d''(x_0) = 0, \quad d'x_0 = -d''x_1, \dots, \quad d'x_{r-2} = -d''x_{r-1}$$

Moreover, if this is the case then $d_r[x_0] = [d'x_{r-1}]$ (1 point).

Problem 8. Let X be a topological space. Show that its singular homology with coefficients in \mathbb{Z} and in \mathbb{Z}_m are related as follows: one has an exact sequence

$$0 \rightarrow H_n^{sing}(X, \mathbb{Z}) \otimes \mathbb{Z}_m \rightarrow H_n^{sing}(X, \mathbb{Z}_m) \rightarrow \text{Tor}_1^{\mathbb{Z}}(H_{n-1}^{sing}(X, \mathbb{Z}), \mathbb{Z}_m) \rightarrow 0$$

(2 points)

Problem 9. Let X^\bullet and Y^\bullet be complexes of R -modules concentrated in non-negative degrees. Their tensor product $(X \otimes_R Y)^\bullet$ is a total complex of a double complex $K^{pq} = X^p \otimes Y^q$ with differentials $d_X \otimes \text{Id}_Y$ and $(-1)^p \text{Id}_X \otimes d_Y$. (a) Show that this is a double complex. (b) Show that if $R = k$ is a field then the spectral sequence of the double complex degenerates at E_2 and one has (not canonically) $H^n(X \otimes Y) \simeq \bigoplus_{p+q=n} H^p(X) \otimes H^q(Y)$. (c) Let X and Y be manifolds. Show that

$$H_{dR}^n(X \times Y, \mathbb{R}) \simeq \bigoplus_{p+q=n} H_{dR}^p(X, \mathbb{R}) \otimes H_{dR}^q(Y, \mathbb{R})$$

(2 points).

Problem 10. Show that Koszul complex $K(x_1, \dots, x_n)$ is isomorphic to the tensor product of complexes $K(x_1) \otimes_R \dots \otimes_R K(x_n)$ (1 point).

Problem 11. Consider a fibration $S^{2n+1} \rightarrow \mathbb{C}\mathbb{P}^n$,

$$\left\{ (z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid \sum_i |z_i|^2 = 1 \right\} \rightarrow \{[z_0 : \dots : z_n] \in \mathbb{P}^n\}.$$

Use Leray spectral sequence to compute de Rham cohomology of $\mathbb{C}\mathbb{P}^n$ (1 point).

Problem 12. Using the action of SU_n on S^{2n-1} , compute de Rham cohomology of $\text{SU}_2, \text{SU}_3, \text{SU}_4$ (1 point).

Problem 13. Let M be a simply-connected manifold. Let $E \rightarrow M$ be an S^r -bundle. Use the Leray spectral sequence to deduce the following exact sequence:

$$\dots \rightarrow H_{dR}^n(E, \mathbb{R}) \rightarrow H_{dR}^{n-r}(M, \mathbb{R}) \rightarrow H_{dR}^{n+1}(M, \mathbb{R}) \rightarrow H_{dR}^{n+1}(E, \mathbb{R}) \rightarrow \dots$$

(1 point)

Problem 14. (a) Show that all pages E_r of the Leray spectral sequence admit an associative multiplication $E_r^{p,q} \times E_r^{p',q'} \rightarrow E_r^{p+p',q+q'}$ such that d_r is an antiderivation. (b) Describe the multiplicative structure on $H_{dR}^\bullet(\mathbb{C}\mathbb{P}^n, \mathbb{R})$. (c) Describe the multiplicative structure on $H_{dR}^\bullet(\text{SU}_n, \mathbb{R})$ (2 points).

Problem 15. Let X be a simply-connected topological space with a base point $*$. Let $P(X)$ (resp. $\Omega(X)$) be the path space (resp. the loop space), i.e. the space of all continuous maps $f : [0, 1] \rightarrow X$ such that $f(0) = *$ (resp. $f(0) = f(1) = *$). Notice that we have a fibration $P(X) \rightarrow X$ with a fiber over $*$ given by $\Omega(X)$. Even though these spaces are not manifolds, there still exists the Leray spectral sequence for singular cohomology (don't prove that). Compute $H_{sing}^*(\Omega(S^2), \mathbb{Z})$ as a ring (2 points).

Problem 16. A covariant δ -functor $T^i : \mathcal{A} \rightarrow \mathcal{B}$ is called *universal* if, given any other δ -functor T' , any natural transformation $f^0 : T^0 \rightarrow T'^0$ can be extended to a unique sequence of natural transformations $f^i : T^i \rightarrow T'^i$, which commute with δ for any short exact sequence. Grothendieck proved

that T is universal if each T^i for $i > 0$ satisfies the following condition: for each object A of \mathcal{A} , there exists a monomorphism $u : A \rightarrow M$ such that $T^i u = 0$. Using Grothendieck's theorem, show that right derived functors of any left exact functor form a universal δ -functor (2 points).

Problem 17. Show that the spectral sequence of a filtered complex K induces the following 5-term exact sequence

$$0 \rightarrow E_2^{1,0} \rightarrow H^1(K) \rightarrow E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \rightarrow H^2(K)$$

assuming all pages of the spectral sequence are in the first quadrant (1 point).

Problem 18. Let $M \subset \mathbb{C}\mathbb{P}_{z_0, \dots, z_n}^n \times \mathbb{C}\mathbb{P}_{w_0, \dots, w_n}^n$ be a hypersurface $\sum_{i=0}^n z_i w_i = 0$. Find its Betti numbers (i.e. dimensions of de Rham cohomology groups) (1 point).

Problem 19. Let $R \rightarrow S$ be a ring homomorphism, let A be an S -module and let B be an R -module. (a) Show that there exists a spectral sequence with an $E_2^{p,q}$ term given by $\text{Ext}_S^p(A, \text{Ext}_R^q(S, B))$ that abuts to $\text{Ext}_R^{p+q}(A, B)$. Hint: use a projective resolution of A and an injective resolution of B to create a double complex. (b) Similarly, show that there is a spectral sequence $\text{Tor}_p^S(\text{Tor}_q^R(S, B), A) \Rightarrow \text{Tor}_{p+q}^R(B, A)$. (2 points).

Problem 20. Let (K^\bullet, d) be any cochain complex. Consider its filtration given by $K \supset \text{Ker } d \supset \text{Im } d$. Describe the corresponding spectral sequence, and in particular show that $d_3 = 0$ (1 point).

Problem 21. Fix an integer p and consider a complex K given by

$$0 \rightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(p,0)} \mathbb{Z} \rightarrow 0.$$

K has an endomorphism $f : K \rightarrow K$ given by multiplication by p . Define a filtration of K by $F^n K = \text{Im}(f^n)$. Work out all pages of the corresponding spectral sequence (2 points).

Problem 22. (a) Compute de Rham cohomology with complex coefficients of $\mathbb{C}\mathbb{P}^1 \setminus \{p_1, \dots, p_n\}$ (n distinct points). (b) Let $M_{0,n}$ be the orbit space

$$\{(p_1, \dots, p_n) \in (\mathbb{C}\mathbb{P}^1)^n \mid p_i \neq p_j\} / \text{PGL}_2.$$

Show that $M_{0,n}$ is isomorphic to a Zariski open subset in \mathbb{C}^{n-3} . (c) Show that the Leray–Hirsch theorem can be applied to the fibration $M_{0,n} \rightarrow M_{0,n-1}$ given by forgetting the last point. What is the fiber of this fibration? (d) Show that $\dim_{\mathbb{C}} H_{dR}^{n-3}(M_{0,n}, \mathbb{C}) = (n-2)!$ (3 points).

HOMEWORK 5. DEADLINE: APRIL 13.

Problem 1. Give an example of a morphism of sheaves $\mathcal{F} \rightarrow \mathcal{G}$ such that the image presheaf is not a sheaf (1 point).

Problem 2. Show that the sheaf can be defined on the basis of topology as discussed in class (1 point).

Problem 3. Show on examples that (a) the functor of global sections of sheaves; (b) the pushforward of sheaves are not exact functors (1 point).

Problem 4. Finish the proof of the theorem that the category of vector bundles of rank n and the category of locally free sheaves of rank n (on complex manifolds) are equivalent (1 point).

Problem 5. (a) Given a morphism of sheaves $\mathcal{F} \rightarrow \mathcal{G}$, show that (a) the kernel presheaf is a sheaf; (b) the kernel sheaf is zero if and only if \mathcal{F}_x injects into \mathcal{G}_x for any $x \in X$. (c) Show that the complex of sheaves is exact if and only if it is exact on stalks. (2 points).

Problem 6. Let $\pi : E \rightarrow B$ be a sphere bundle, i.e. a fiber bundle with fiber S^k . Let's call π orientable if there exists an atlas $\{U_\alpha\}$ and trivializations $\tau_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times S^n$ such that transition functions $g_{\alpha\beta} = \tau_\beta \tau_\alpha^{-1}$ on overlaps induce orientation-preserving diffeomorphisms of fibers. (a) Show that π is orientable if and only if the monodromy representation is trivial. (b) Show that if $\pi_1(B)$ has no subgroups of index 2 then π is orientable. (c) Show that if π is orientable then there exists a class $e \in H_{dR}^{k+1}(B, \mathbb{R})$, called an *Euler class* such that one has the following exact sequence

$$\dots \rightarrow H_{dR}^n(E, \mathbb{R}) \rightarrow H_{dR}^{n-k}(M, \mathbb{R}) \xrightarrow{e \wedge} H_{dR}^{n+1}(M, \mathbb{R}) \rightarrow H_{dR}^{n+1}(E, \mathbb{R}) \rightarrow \dots$$

(2 points).

Problem 7. Let $L \rightarrow B$ be a complex line bundle over a complex manifold B . Viewing L as a rank 2 real vector bundle, remove the zero-section and take the quotient $\mathbb{R}^2 \setminus \{0\} \rightarrow S^1$ of each fiber by the scalar action of the multiplicative group \mathbb{R}_+ . (a) Show that this gives an orientable S^1 -bundle $E \rightarrow B$. Its Euler class $e \in H_{dR}^2(B, \mathbb{R})$ is called the *Chern class* of L . (b) Let $\mathcal{O}(-1)$ be the line bundle over $\mathbb{C}P^1$ defined as follows: its fiber over $p \in \mathbb{C}P^1$ is the complex line in \mathbb{C}^2 that corresponds to p . Find a 2-form on $\mathbb{C}P^1$ that represents the Chern class of $\mathcal{O}(-1)$ (2 points).

Problem 8. Show that a constant sheaf of stalk \mathbb{Z} is a sheafification of a constant presheaf of stalk \mathbb{Z} (1 point).

Problem 9. Show that a morphism of sheaves is an isomorphism if and only if it is both injective and surjective (1 point).

Problem 10. (a) Give a definition for a subsheaf \mathcal{F} of \mathcal{G} and for a quotient sheaf \mathcal{G}/\mathcal{F} . (b) Let $\phi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves. Show that $\text{Im } \phi \simeq \mathcal{F}/\text{Ker } \phi$ and $\text{Coker } \phi \simeq \mathcal{G}/\text{Im } \phi$ (1 point).

Problem 11. Let $\{\mathcal{F}_i\}$ be a direct system of sheaves and morphisms of sheaves. (a) Show that $U \mapsto \varinjlim \mathcal{F}_i(U)$ is a presheaf. (b) Show that its sheafification is a direct limit in the category of sheaves (i.e. satisfies the universal property of a direct limit) (1 points).

Problem 12. Let \mathcal{F} be the presheaf of part (a) of the previous problem. (a) Show that \mathcal{F} is not necessarily a sheaf. (b) Show that \mathcal{F} is a sheaf if X is a Noetherian topological space, i.e. its closed subsets satisfy the descending chain condition. (3 points).

Problem 13. Let $s \in \mathcal{F}(U)$ be a local section of a sheaf \mathcal{F} . Its support $\text{Supp}(s)$ is defined as a locus of points $p \in U$ such that $s_p \neq 0$. Show that $\text{Supp}(s)$ is closed (1 point).

Problem 14. Let \mathcal{F}, \mathcal{G} be sheaves on X . (a) Show that the set of morphisms $\text{Hom}(\mathcal{F}, \mathcal{G})$ is an Abelian group. (b) Show that $U \mapsto \text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ is a sheaf, called the sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ of local morphisms. (c) Show that $\text{Hom}(\mathcal{F}, \cdot)$ (resp. $\mathcal{H}om(\mathcal{F}, \cdot)$) is a left-exact functor from the category of sheaves on X to the category of Abelian groups (resp. sheaves on X). (2 points).

Problem 15. Let X be a connected topological space. Show that the constant sheaf of stalk \mathbb{Z} is flasque if and only if X is an irreducible topological space, i.e. X can not be expressed as a union of two proper closed subsets (1 point).

Problem 16. Is it true that the quotient of a flasque sheaf by a flasque subsheaf is flasque (1 point)?

Problem 17. Let \mathcal{F} be a sheaf which is injective as an object in the category of sheaves. Show that \mathcal{F} is flasque (2 points).

Problem 18. Show that the structure sheaf \mathcal{O} of holomorphic functions on \mathbb{C} is not flasque (1 point).

Problem 19. Let $f : X \rightarrow Y$ be a continuous map of topological spaces. Let \mathcal{G} be a sheaf on Y . The inverse image sheaf $f^{-1}(\mathcal{G})$ is the sheafification of the presheaf $U \mapsto \varinjlim_{f(U) \subset V} \mathcal{G}(V)$, where the direct limit is over all open subsets of Y containing $f(U)$. Show that this gives a functor from sheaves on Y to sheaves on X , which is a left adjoint functor of the pushforward functor f_* from sheaves on X to sheaves on Y (2 points).

Problem 20. Let $\dots \mathcal{F}_2 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_0 \rightarrow 0$ be an exact sequence of sheaves. Show that there exists a spectral sequence in the second quadrant with $E_1^{pq} = H^q(X, \mathcal{F}_{-p})$ which abuts to 0 (2 points).

HOMEWORK 6. DEADLINE: MAY 4.

We will have a final exam (of sorts): along with Homework 6 you can submit any problems from the previous homeworks worth 2 or more points. There will be no partial credit, i.e. all parts have to be solved. However, you can finish problems that you have already partially submitted for the rest of the credit.

Problem 1. Let X be a manifold with covering $\{U_i\}$ such that all finite intersections of U_i 's are contractible. Show that there exists a double complex with

$$C^{pq} = \bigoplus_{i_0 < \dots < i_p} \mathcal{A}^q(U_{i_0} \cap \dots \cap U_{i_p})$$

where horizontal differentials are Čech differentials and vertical differentials are de Rham differentials. Conclude that $H_{dR}^n(X, \mathbb{C}) \simeq H_{\mathcal{U}}^n(X, \mathbb{C})$. (1 point).

Problem 2. Let L be a complex line bundle on a smooth manifold X with transition functions $g_{\alpha\beta}$ with respect to a sufficiently fine open covering $\mathcal{U} = \{U_\alpha\}$. Let ρ_α be a partition of unity with respect to \mathcal{U} . Show that the first Chern class $c_1(L) \in H^2(X, \mathbb{Z})$ is represented by the following closed 2-form in $H_{dR}^2(X, \mathbb{C})$:

$$\omega|_{U_\alpha} = \frac{1}{2\pi i} \sum_{\gamma} d(\rho_\gamma d \log g_{\alpha\gamma}).$$

(2 points).

Problem 3. Let X be a complex Riemann surface and let $D = \sum_{i=1}^k a_i p_i$ be a divisor on X , i.e. a collection of points $p_1, \dots, p_k \in X$ with multiplicities. Let $\mathcal{O}_X(D)$ be a sheaf on X defined as follows: for any open subset $U \subset X$, $\mathcal{O}_X(D)(U)$ is the set of meromorphic functions f on U holomorphic outside of $\{p_1, \dots, p_k\}$ and such that near each p_i with can use a holomorphic coordinate z to write $f = z^{-a_i} g$, where g is holomorphic. (a) Show that $\mathcal{O}_X(D)$ is a sheaf of holomorphic sections of a holomorphic line bundle L . (b) Let $\omega \in H_{dR}^2(X, \mathbb{C})$ be a form representing the first Chern class of L . Use the Stokes formula to show that

$$\int_X \omega = \sum_{i=1}^k a_i.$$

(Hint: it is will be convenient not to use the output of the previous problem directly, but rather to tinker with its proof a little bit). (2 points).

Problem 4. Is the category of presheaves abelian? Find the right-adjoint of the sheafification functor. Is sheafification (left,right) exact? (1 point).

Problem 5. Let X be a C^∞ manifold. Show that real line bundles on X are classified (up to an isomorphism) by elements of the group $H^1(X, \mathbb{Z}_2)$ (1 point).

Problem 6. Consider $X = \mathbb{CP}^1$ with Zariski topology (i.e. a proper subset is closed if and only if it is finite). Let \mathcal{K} be a field of rational functions in one variable. We view \mathcal{K} as a constant sheaf on \mathbb{CP}^1 . (a) Show that X carries a structure sheaf \mathcal{O}_X defined as follows: $\mathcal{O}_X(U) \subset \mathcal{K}$ is a subring which consists of functions without poles in U . (b) Show that $0 \rightarrow \mathcal{O}_X \rightarrow$

$\mathcal{K} \rightarrow \mathcal{K}/\mathcal{O}_X \rightarrow 0$ is a flasque resolution of \mathcal{O}_X . (c) Use part (b) to show that $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$. (2 points).

Problem 7. In the set-up of the previous problem, show that there is no sheaf of \mathcal{O}_X -modules \mathcal{P} with surjection $\mathcal{P} \rightarrow \mathcal{O}_X$ such that \mathcal{P} is a projective object in the category of \mathcal{O}_X -modules (2 points).

Problem 8. Let R be a commutative ring. Recall that an affine scheme $\text{Spec } R$ is the set of prime ideals of R endowed with Zariski topology: closed subsets $Z \subset \text{Spec } R$ are subsets of the form $V(I) = \{\mathfrak{p} \in \text{Spec } R \mid \mathfrak{p} \supset I\}$ for various ideals $I \subset R$. (a) Show that Zariski topology has a basis of *principal open sets* $D_f = \{\mathfrak{p} \in \text{Spec } R \mid f \notin \mathfrak{p}\}$ for $f \in R$. (b) Show that $X = \text{Spec } R$ carries a structure sheaf \mathcal{O}_X with the following property: $\mathcal{O}_X(D_f) = R_f$ for any $f \in R$, where $R_f = R[1/f]$ is the localization of R in the multiplicative system $\{1, f, f^2, f^3 \dots\}$. Moreover, the stalk of \mathcal{O}_X at $\mathfrak{p} \in \text{Spec } R$ is the local ring $R_{\mathfrak{p}}$. (2 points).

Problem 9. Consider the category of filtered abelian groups: objects are abelian groups A along with an increasing filtration $\dots \subset A^i \subset A^{i+1} \subset \dots \subset A$ and morphisms are homomorphisms preserving filtration. Show that this category is not abelian (1 point).

Problem 10. (a) Show that the push-forward of a flasque sheaf is flasque. (b) Let $i : Y \hookrightarrow X$ be a closed subset of a topological space (with induced topology) and let \mathcal{F} be a sheaf on Y . Show that $H^k(Y, \mathcal{F})$ is isomorphic to $H^k(X, i_*\mathcal{F})$ (2 points).

Problem 11. Let \mathcal{F} be a flasque sheaf on a topological space. Let $\mathcal{U} = \{U_\alpha\}$ be an arbitrary covering. Show that Čech cohomology groups $H_{\mathcal{U}}^i(X, \mathcal{F})$ vanish for $i > 0$. (1 point).

In problems 12–15, (X, \mathcal{O}_X) is an arbitrary ringed space.

Problem 12. Show that the category of sheaves of \mathcal{O}_X -modules is abelian and has enough injectives (1 point).

Problem 13. A sheaf \mathcal{F} of \mathcal{O}_X -modules is said to be *generated by global sections* if, for any point $x \in X$, there exist global sections $s_i \in \Gamma(X, \mathcal{F})$, $i \in I$, such that the stalk \mathcal{F}_x is generated by germs $(s_i)_x$ as an $\mathcal{O}_{X,x}$ -module. Show that \mathcal{F} is generated by global sections if and only if \mathcal{F} is isomorphic to a quotient sheaf of a free sheaf (1 point).

Problem 14. Let \mathcal{F}, \mathcal{G} be sheaves of \mathcal{O}_X -modules. Show that one has a presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{G}(U)$, which however is not always a sheaf. Its sheafification is denoted by $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ (2 points).

Problem 15. Let \mathcal{F} be a sheaf of \mathcal{O}_X -modules. We define $\text{Ext}^i(\mathcal{F}, \cdot)$ as a right derived functor of $\text{Hom}(\mathcal{F}, \cdot)$ and we define $\mathcal{E}xt^i(\mathcal{F}, \cdot)$ as a right derived functor of $\mathcal{H}om(\mathcal{F}, \cdot)$ (cf. Exercise 14 of the previous homework). (a) Show that $\mathcal{E}xt^0(\mathcal{O}_X, \mathcal{G}) = \mathcal{G}$, $\mathcal{E}xt^i(\mathcal{O}_X, \mathcal{G}) = 0$ for $i > 0$. (b) Show that $\text{Ext}^i(\mathcal{O}_X, \mathcal{G}) \simeq H^i(X, \mathcal{G})$. (c) Let $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ be an exact sequence of sheaves. Deduce the following long exact sequences

$$\dots \rightarrow \text{Ext}^i(\mathcal{F}'', \mathcal{G}) \rightarrow \text{Ext}^i(\mathcal{F}, \mathcal{G}) \rightarrow \text{Ext}^i(\mathcal{F}', \mathcal{G}) \rightarrow \text{Ext}^{i+1}(\mathcal{F}'', \mathcal{G}) \rightarrow \dots$$

and

$$\dots \rightarrow \mathcal{E}xt^i(\mathcal{F}'', \mathcal{G}) \rightarrow \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \rightarrow \mathcal{E}xt^i(\mathcal{F}', \mathcal{G}) \rightarrow \mathcal{E}xt^{i+1}(\mathcal{F}'', \mathcal{G}) \rightarrow \dots$$

(2 points).

Problem 16. Let \mathcal{A} be an abelian category with enough injectives. Show that any cochain complex A^\bullet of objects of \mathcal{A} has a *Cartan–Eilenberg resolution*: a double complex $I^{\bullet,\bullet}$ in the upper-half plane such that all $I^{p,q}$ are injective; vertical cohomology groups vanish for $q > 0$ and are equal to A^p for $q = 0$; p -th column of horizontal coboundaries gives an injective resolution of coboundary in A^p ; p -th column of horizontal cohomologies gives an injective resolutions of $H^p(A)$. (1 point).

Problem 17. Let $G : \mathcal{A} \rightarrow \mathcal{B}, F : \mathcal{B} \rightarrow \mathcal{C}$ be left-exact covariant functors of abelian categories with enough injectives. Suppose G takes injective objects of \mathcal{A} into F -acyclic objects of \mathcal{B} . Use the previous exercise to show that for any object X of \mathcal{A} there exists a spectral sequence with

$$E_2^{p,q} = R^q F(R^p G(X)) \Rightarrow R^{p+q}(FG)(X)$$

(Grothendieck's spectral sequence of a composite functor). (2 points).

Problem 18. Let $\pi : X \rightarrow Y$ be a continuous map of topological spaces and let \mathcal{F} be a sheaf on X . Show that there exists a *Leray spectral sequence* with

$$E_2^{p,q} = H^q(Y, R^q \pi_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}),$$

where $R^q \pi_*$ is the right-derived functor of the push-forward. (2 points).

Problem 19. Let \mathcal{F}, \mathcal{G} be sheaves of \mathcal{O}_X -modules on a ringed space (X, \mathcal{O}_X) . Show that there exists a *local-to-global* spectral sequence with

$$E_2^{p,q} = H^q(X, \mathcal{E}xt^p(\mathcal{F}, \mathcal{G})) \Rightarrow \text{Ext}^{p+q}(\mathcal{F}, \mathcal{G})$$

(2 points).

Problem 20. Let G be a group and let A be a G -module. We define *group cohomology*

$$H^n(G, A) = \text{Ext}_{\mathbb{Z}[G]}^n(\mathbb{Z}, A),$$

where \mathbb{Z} is a trivial G -module. (a) Show that $H^n(G, A)$ computes the right derived functor of a left-exact covariant functor $A \mapsto A^G$ (from the category of G -modules to the category of Abelian groups). (b) Let $G = \mathbb{Z}$ and let $A = \mathbb{Z}$ be a trivial G -module. Compute $H^n(G, A)$ (2 points).

Problem 21. Let H be a normal subgroup of a group G . Let A be a G -module. Show that there exists a *Hochschild–Serre* spectral sequence with

$$E_2^{p,q} = H^q(G/H, H^p(H, A)) \Rightarrow H^{p+q}(G, A).$$

(2 points).

Problem 22. Let $\pi : E \rightarrow B$ be a fiber bundle with fiber F in the category of smooth manifolds. We define a sheaf $R^k \pi_*(E, \mathbb{R})$ on B as a sheaf associated to a presheaf $U \mapsto H_{dR}^k(\pi^{-1}(U), \mathbb{R})$. (a) Show that $R^k \pi_*(E, \mathbb{R})$ is locally constant with stalk $H_{dR}^k(F, \mathbb{R})$, i.e. there exists a covering $\{U_i\}$ of B such that $R^k \pi_*(E, \mathbb{R})|_{U_i}$ is a constant sheaf of stalk $H_{dR}^k(F, \mathbb{R})$. (b) Show that $R^k \pi_*(E, \mathbb{R})$ is a constant sheaf for any k if and only if there is no monodromy. (1 point).

Problem 23. Let \mathcal{F} be a sheaf on a topological space X . Consider the set of all coverings of X as a partially ordered set ordered by refinement. (a) Show that Čech cohomology $\{H_{\mathcal{U}}^\bullet(X, \mathcal{F})\}_{\mathcal{U}}$ forms a direct system. Let

$$H_{\text{Čech}}^\bullet(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} \{H_{\mathcal{U}}^\bullet(X, \mathcal{F})\}_{\mathcal{U}}.$$

(b) Show that $H^\bullet(X, \mathcal{F}) \simeq H_{\check{C}ech}^\bullet(X, \mathcal{F})$ (2 points).