§4. **Weighted projective spaces (Mar 2, 4, 7)**

§4.1. **First examples.** Let's move from quotient spaces by finite groups to the actions of the easiest infinite group: $\mathbb{C}^*$. We will work out the following extremely useful example: fix positive integers $a_0, \ldots, a_n$ (called weights) and consider the action of $\mathbb{C}^*$ on $\mathbb{A}^{n+1}$ defined as follows:

$$\lambda \cdot (x_0, \ldots, x_n) = (t^{a_0}x_0, \ldots, t^{a_n}x_n) \quad \text{for any} \quad t \in \mathbb{C}^*.$$

The quotient (which we are going to construct) is called the weighted projective space. Notation:

$$\mathbb{P}(a_0, \ldots, a_n).$$

For example, we have

$$\mathbb{P}(1, \ldots, 1) = \mathbb{P}^n.$$

4.1.1. EXAMPLE. We have met $\mathbb{P}(4,6)$ before. Recall that any elliptic curve has a Weierstrass equation $y^2 = 4x^3 - g_2x - g_3$, $\Delta = g_2^3 - 27g_3^2 \neq 0$ and this is an extremely useful fact for studying elliptic fibrations (and elliptic curves defined over rings of algebraic integers). Coefficients g_2 and g_3 are defined not uniquely but only up to admissible transformations

$$g_2 \mapsto t^4 g_2, \quad g_3 \mapsto t^6 g_3.$$

So the moduli space of elliptic curves is $\mathbb{P}(4,6)$ with a point "at infinity" removed (which corresponds to the $\mathbb{C}^*$-orbit $\{\Delta = 0\}$). So it should come at no surprise that

$$\mathbb{P}(4,6)_{[g_2:g_3]} \simeq \mathbb{P}^1_{[j:1]},$$

where j is given by the usual formula (2.7.4). Notice however that thinking about $\mathbb{P}(4,6)$ has a lot of advantages: it encompasses the idea of Weierstrass families better and it emphacizes the role of special elliptic curves with many automorphisms. In general, we will see that weighted projective spaces are different from usual ones: they have singularities.

4.1.2. EXAMPLE. Let's construct $\mathbb{P}(1,1,2)$ by hand. Take the map

$$\pi : \mathbb{A}^3_{x,y,z} \setminus \{0\} \to \mathbb{P}^3_{[A:B:C:D]}, \quad (x,y,z) \mapsto [x^2 : xy : y^2 : z].$$

It is easy to see that it separates orbits, i.e. $\pi(x,y,z) = (x',y',z')$ if and only if there exists $t \in \mathbb{C}^*$ such that

$$x' = tx, \ y' = ty, \ z' = t^2 z.$$

The image is a quadratic cone $AB = C^2$ in $\mathbb{P}^3$.

Now let's discuss the general construction of the weighted projective space. Remember the drill: we have to find all semi-invariants. Here this is exceptionally easy: any monomial $x_0^{i_0} \ldots x_n^{i_n}$ is a semi-invariant for the $\mathbb{C}^*$-action of weight $w = i_0a_0 + \ldots + i_na_n$ (i.e. $t \in \mathbb{C}^*$ acts by multiplying this monomial by t^w). So the algebra of semi-invariants is just the full polynomial algebra

$$\mathbb{C}[x_0, \ldots, x_n].$$

However, we have to introduce a different grading on this algebra, where each variable x_i has degree a_i. Here are some basic observations:

- There are no non-constant invariants. So we can not produce a quotient by our method of taking $\mathrm{MaxSpec}$ of the algebra of invariants (by taking the image of the map to $\mathbb{A}^r$ given by r basic invariants). Here is an "explanation": all orbits contain zero in their closure. So any invariant polynomial is just a constant equal to the value of this polynomial in 0. This is the reason we have to remove zero, just like in the $\mathbb{P}^n$ case. Notice that $\mathbb{A}^{n+1}\setminus\{0\}$ is not an affine variety anymore. The procedure of taking $\mathrm{MaxSpec}$ won't work after removing the origin.
- The algebra of semi-invariants is generated by variables, which have different degrees. So the situation is different from our experience of writing the Grassmannian $G(2,n)$ as a quotient $\mathrm{Mat}(2,n)/GL_2$, where basic semi-invariants (2×2 minors) all had the same degree.

So we need a new approach. The idea is simple: $\mathbb{A}^{n+1}\setminus\{0\}$ is covered by principal open sets $D(x_i)$. We will take take their quotients by $\mathbb{C}^*$ first and then glue them, just like in the definition of the usual projective space.

In the case of $\mathbb{P}^n$ we don't even notice the $\mathbb{C}^*$ action because we kill it by setting $x_i = 1$. So we quite naturally identify $D(x_i)/\mathbb{C}^* \simeq \mathbb{A}^n$. Let's denote the corresponding chart $D_{x_i}\subset\mathbb{P}^n$ to distinguish it from $D(x_i)\subset\mathbb{A}^{n+1}$. What will happen in a more general case? Setting $x_i = 1$ does not quite eliminate t: it just implies that $t^{a_i} = 1$. This is still an achievement: it shows that the action of $\mathbb{C}^*$ on $D(x_i)\subset\mathbb{A}^{n+1}$ is reduced to the action of μ_{a_i} on $\mathbb{A}^n$. This is a familiar ground: the quotient will be

$$D_{x_i} = D(x_i)/\mathbb{C}^* \simeq \mathbb{A}^n/\mu_{a_i} = \mathrm{MaxSpec}\,\mathbb{C}[x_0,\ldots,\hat{x}_i,\ldots,x_n]^{\mu_{a_i}},$$

where μ_{a_i} acts with weights $a_0,\ldots\hat{a}_i,\ldots,a_n$. So for example, a projective quadratic cone $\mathbb{P}(1,1,2)$ is covered by three charts: two copies of $\mathbb{A}^2$ and one copy of $\frac{1}{2}(1,1)$, which is isomorphic to an affine quadratic cone.

Here is another way of thinking about this. Notice that

$$\mathcal{O}(D(x_i)) = \mathbb{C}\left[x_0,\ldots,x_n;\ \frac{1}{x_i}\right]$$

and that $\mathbb{C}^*$ now acts on the affine variety $D(x_i)$. We can use our old recipe for computing the quotient: take the algebra of invariants and compute its spectrum. So we set

$$\mathcal{O}(D_{x_i}) = \mathcal{O}(D(x_i))^{\mathbb{C}^*} = \left\{\frac{p}{x_i^k}\quad\Big|\quad p\in\mathbb{C}[x_0,\ldots,x_n],\quad \deg p = ka_i\right\}$$

(here and after the degree $\deg$ is our funny weighted degree). There are two cases: if $a_i = 1$ then we just have

$$\mathcal{O}(D_{x_i}) = \mathbb{C}\left[\frac{x_0}{x_i^{a_0}},\ldots,\frac{x_n}{x_i^{a_n}}\right]\simeq\mathbb{C}[y_1,\ldots,y_n].$$

The chart is an affine space, just like for the standard $\mathbb{P}^n$. To figure out the general case, for simplicity let's restrict to the weighted projective plane $\mathbb{P}(a_0,a_1,a_2)$. What will be the first chart? Consider the cyclic field extension

$$\mathbb{C}(x_0,x_1,x_2)\subset\mathbb{C}(z_0,x_1,x_2),$$

where $x_0 = z_0^{a_0}$. Then we have

$$\mathcal{O}(D_{x_i}) = \left\{ \frac{p}{z_0^{a_0 k}} \quad | \quad p \in \mathbb{C}[x_0, x_1, x_2], \quad \deg p = k a_0 \right\} =$$

$$\left\{ \sum_{i,j} a_{ij} \left(\frac{x_1}{z_0^{a_1}}\right)^i \left(\frac{x_2}{z_0^{a_2}}\right)^j \quad | \quad a_1 i + a_2 j \equiv 0 \mod a_0 \right\} \subset \mathbb{C}\left[\frac{x_1}{z_0^{a_1}}, \frac{x_2}{z_0^{a_2}}\right].$$

So we get a subalgebra in $\mathbb{C}[y_1, y_2]$ spanned by monomials $y_1^i y_2^j$ such that $a_1 i + a_2 j \equiv 0 \mod a_0$. This is our old friend, the cyclic quotient $\frac{1}{a_0}(a_1, a_2)$.

§4.2. **Proj (projective spectrum).** Let's generalize this even further. Let R be any finitely generated graded integral domain such that $R_0 = \mathbb{C}$. We can write R as a quotient of $\mathbb{C}[x_0, \ldots, x_n]$ (with grading given by degrees $a_0, \ldots, a_n$ of homogeneous generators of R) by a homogeneous (in this grading!) prime ideal. Functions in this ideal are constant along $\mathbb{C}^*$ orbits in $\mathbb{A}^{n+1}$. As a set, we simply define

$$\operatorname{Proj} R \subset \mathbb{P}(a_0, \ldots, a_n)$$

as a set of $\mathbb{C}^*$-orbits where all functions in the ideal vanish.

Rational functions on $\operatorname{Proj} R$ are defined as ratios of polynomials of the same (weighted) degree, i.e.

$$\mathbb{C}(\operatorname{Proj} R) = (\operatorname{Quot} R)_0,$$

where the subscript means that we are only taking fractions of degree 0. We call a function regular at some point if it has a presentation as a fraction with a denominator non-vanishing at this point. It is clear that $\operatorname{Proj} R$ is covered by affine charts D_f for each homogeneous element $f \in R$ of positive degree, where

$$\mathcal{O}(D_f) = R[1/f]_0.$$

What is the gluing? Given D_f and D_g, notice that

$$D_f \cap D_g = D_{fg},$$

is a principal open subset in both D_f (where it is a complement of a vanishing set of a regular function $\frac{g^{\deg f}}{f^{\deg g}}$) and D_g (where we use $\frac{f^{\deg g}}{g^{\deg f}}$). Formally speaking, we have to check that in $\mathbb{C}(\operatorname{Proj} R)$ we have

$$R[1/fg]_0 = R[1/f]_0 \left[\frac{f^{\deg g}}{g^{\deg f}}\right]. \tag{4.2.1}$$

This kind of formulas are proved by tinkering with fractions with a sole purpose to balance degrees of the numerator and the denominator. We leave it as an exercise.

§4.3. **Abstract algebraic varieties.** To continue this discussion, we have to ask ourselves: what is it that we are trying to prove? We will later see that $\operatorname{Proj} R$ is in fact a projective variety, but at this point it would be useful to give a definition of an abstract algebraic variety.

4.3.1. DEFINITION. For simplicity, we will only define an *irreducible algebraic variety* X. We need

- A finitely generated field extension K of $\mathbb{C}$. This will be a field of rational functions on X.
- Topology on X.
- For each open subset $U \subset X$ we need a subalgebra $\mathcal{O}_X(U) \subset K$. It should satisfy the condition
$$\mathcal{O}_X\left(\bigcup_{i\in I} U_i\right) = \bigcap_{i\in I} \mathcal{O}_X(U_i).$$
$\mathcal{O}_X$ is called the *structure sheaf*.
- Finally, X should admit a finite cover $\{U_i\}$ such that each U_i (with an induced topology) is an irreducible affine variety (with Zariski topology) with function field K and for each open subset $V \subset U_i$, $\mathcal{O}_X(V) \subset K$ is the algebra of rational functions regular on V.

In practice, algebraic varieties are constructed by gluing affine varieties. Suppose A and B are irreducible affine varieties with the same function field K. Suppose, in addition, that there exists another affine variety C and open immersions

$$i_A : C \hookrightarrow A \quad \text{and} \quad i_B : C \hookrightarrow B.$$

Then we define the topological space $X = A \cup_C B$ by identifying points $i_A(x)$ with $i_B(x)$ for any $x \in C$ and by declaring a subset $U \subset X$ open if $U \cap A$ and $U \cap B$ is open. Finally, we set

$$\mathcal{O}_X(U) = \mathcal{O}_A(U \cap A) \cap \mathcal{O}_B(U \cap B)$$

It is easy to generalize this to several affine charts: we need irreducible affine varieties

$$U_0, \ldots, U_r,$$

with the same function field. For each pair U_i, U_j we have affine open subsets

$$U_{ij} \subset U_i, \quad U_{ji} \subset U_j$$

and an isomorphism

$$\phi_{ij} : U_{ij} \to U_{ji}.$$

This isomorphism should satisfy (draw pictures)

- $\phi_{ij} = \phi_{ji}^{-1}$,
- $\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and
- $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $U_{ij} \cap U_{ik}$.

4.3.2. LEMMA. $\operatorname{Proj} R$ *is an algebraic variety.*

Proof. We have $K = (\mathrm{Quot}\, R)_0$. For any homogeneous $f \in R$ we have an affine variety

$$D_f = \mathrm{MaxSpec}\, R[1/f]_0.$$

To get a finite atlas, take only homogeneous generators of R. To see the gluing condition, notice that D_{fg} is a principal open subset in both D_f and D_g. The compatibility conditions on triple overlaps are of set-theoretic nature, and are clearly satisfied. □

§4.4. **Separatedness.** There is one annoying phenomenon that we can discuss now and then safely ignore later on. One can take two copies of $\mathbb{A}^1$ and glue them along $D(x) = \mathbb{A}^1 \setminus \{0\}$. This produces a famous "line with two origins" (draw). What's happening here is that diagonally embedded $D(x)$ is not closed in the product of charts (draw), compare with how $\mathbb{P}^1$ is glued (draw). So we give

4.4.1. DEFINITION. An algebraic variety is called *separated* if it has an affine atlas such that for any pair A, B of charts with $C = A \cap B$, the diagonal inclusion of C in $A \times B$ is a closed subset of the product.

How to check this in practice?

4.4.2. LEMMA. *Suppose any two affine charts A and B with $C = A \cap B$ have the following property: there exists $f \in \mathcal{O}(A)$ and $g \in \mathcal{O}(B)$ such that*

$$\mathcal{O}(C) = \mathcal{O}(A)_f = \mathcal{O}(B)_g \subset K.$$

Then X is separated iff for any A and B, we have

$$\mathcal{O}(C) \quad \text{is generated by} \quad \mathcal{O}(A) \quad \textit{and} \quad \mathcal{O}(B).$$

In particular, $\mathrm{Proj}\, R$ *is separated.*

Proof. We have $\mathcal{O}(A \times B) = \mathcal{O}(A) \otimes_k \mathcal{O}(B)$ *(why?)* , and the diagonal map $\Delta : C \to A \times B$ is given by a homomorphism

$$\Delta^* : \mathcal{O}(A) \otimes_k \mathcal{O}(B) \to \mathcal{O}(C), \quad f \otimes g \mapsto \frac{f}{1} \cdot \frac{g}{1}.$$

The closure $\overline{(\Delta(C))}$ of the diagonal is defined by the kernel of Δ^*. In particular, its algebra of functions is $\mathcal{O}(A)\mathcal{O}(B) \subset \mathcal{O}(C)$. So X is separated iff this inclusion is an equality for any pair of charts.

The last remark follows from the formula

$$(R_{fg})_0 = (R_f)_0 (R_g)_0, \tag{4.4.3}$$

which we leave as an exercise. □

§4.5. **Veronese embedding.** We now have two models for $\mathbb{P}(1,1,2)$: as a weighted projective plane defined by charts and as a quadratic cone in $\mathbb{P}^3$. What is the relationship between these models? We are going to show that in fact any $\mathrm{Proj}\, R$ is a projective variety.

4.5.1. DEFINITION. If R is a graded ring then its subring $R^{(d)} = \sum_{d|n} R_n$ is called a d-th Veronese subring.

For example, for $\mathbb{P}(1,1,2)$ the second Veronese subring is generated by x^2, xy, y^2, and z, subject to a single quadratic relation. So $\mathrm{Proj}\, R^{(2)}$ is a quadratic cone in $\mathbb{P}^3$ in this case. The basic fact is:

4.5.2. PROPOSITION. $\operatorname{Proj} R = \operatorname{Proj} R^{(d)}$ *for any* d.

Proof. First of all, we have $(\operatorname{Quot} R)_0 = (\operatorname{Quot} R^{(d)})_0$. Indeed, any fraction $a/b \in (\operatorname{Quot} R)_0$ can be written as $ab^{d-1}/b^d \in (\operatorname{Quot} R^{(d)})_0$.

Let $f_1, \ldots, f_r$ be homogeneous generators of R, so that $\operatorname{Proj} R$ is covered by charts D_{f_i}. Then $f_1^d, \ldots, f_r^d \in R^{(d)}$ are not necessarily generators, however $\operatorname{Proj} R^{(d)}$ is still covered by charts $D_{f_i^d}$. Indeed, if all f_i^d vanish at some point $p \in \operatorname{Proj} R^{(d)}$ then also any function in the ideal generated by them (and hence any function in its radical) vanishes at p. But any generator g of $R^{(d)}$ can be expressed as a polynomial in $f_1, \ldots, f_r$, and therefore a sufficiently high power of g belongs to the ideal (in $R^{(d)}$) generated by $f_1^d, \ldots, f_r^d$. So we have

$$\operatorname{Proj} R = \bigcup_{i=1}^{r} D_{f_i} \quad \text{and} \quad \operatorname{Proj} R^{(d)} = \bigcup_{i=1}^{r} D_{f_i^d}$$

The basic local calculation we need is that charts D_{f_i} of $\operatorname{Proj} R$ and $D_{f_i^d}$ of $\operatorname{Proj} R^{(d)}$ can be identified, i.e. that

$$R^{(d)}[1/f^d]_{(0)} \simeq R[1/f]_{(0)}$$

for any homogeneous element f of R. But indeed,

$$\frac{g}{f_i} = \frac{f^{dj-i}g}{f^{dj}}$$

as soon as $dj > i$. So $\operatorname{Proj} R$ and $\operatorname{Proj} R^d$ have the same charts glued in the same way. $\square$

Now another basic algebraic fact is:

4.5.3. LEMMA. *For a sufficiently large* d, $R^{(d)}$ *is generated by* R_d.

Proof. Let $a_1, \ldots, a_r$ be degrees of homogeneous generators $f_1, \ldots, f_r$ of R. Let $a = \operatorname{l.c.m.}(a_1, \ldots, a_r)$ and let $d = ra$. We claim that this d works. For each i, let $a = a_i b_i$: then

$$\deg f_i^{b_i} = a.$$

Now take any element $f \in R_{kd}$. We claim that it can be written as a polynomial in elements of R_d. It suffices to consider a monomial $f = f_1^{n_1} \ldots f_r^{n_r}$. For inductive purposes, notice that $\deg f = kd = (kr)a$. If $n_i < b_i$ for each i then

$$\deg f < ra = d,$$

a contradiction. So we can write $f = f_i^{b_i} g$, where $\deg g = \deg f - a$. Continuing inductively, we will write

$$f = [f_{i_1}^{b_{i_1}} \ldots f_{i_s}^{b_{i_s}}]g,$$

where $\deg g = d$ and degree of the first term is a multiple of d. Since $\deg f_i^{b_i} = a$ for each i, we can group elements of the first term into groups of r powers each of degree d. This shows that f can be written as a polynomial in elements of R_d. $\square$

By the lemma we can realize $\operatorname{Proj} R$ as a subvariety in $\mathbb{P}^N$ for a sufficiently large N. Indeed, $\operatorname{Proj} R \simeq \operatorname{Proj} R^{(d)}$ and

$$R^{(d)} = \mathbb{C}[y_0, \ldots, y_N]/I,$$

where I is a homogeneous ideal (in the usual sense). So

$$\operatorname{Proj} R^{(d)} = V(I) \subset \mathbb{P}^N.$$

4.5.4. COROLLARY. $\operatorname{Proj} R$ *is a projective variety.*

§5. $\mathbf{M}_2$ (and $\mathbf{A}_2$) - Part I. (Mar 9, 11, 21, 23, 25)

We are going to spend a considerable amount of time studying the moduli space M_2 of algebraic curves of genus 2. Incidentally, this will also give us the moduli space A_2 of *principally polarized Abelian surfaces*: those are algebraic surfaces isomorphic to $\mathbb{C}^2/\Lambda$, where $\Lambda \simeq \mathbb{Z}^4$ is a lattice. So Abelian surfaces are naturally Abelian groups just like elliptic curves. We will see that M_2 embeds in A_2 as an open subset (via the *Jacobian* construction) and the complement $A_2 \setminus M_2$ parametrizes split Abelian surfaces of the form $E_1 \times E_2$, where E_1 and E_2 are elliptic curves. The map $M_g \hookrightarrow A_g$ can be constructed in any genus (its injectivity is called the *Torelli theorem*) but dimensions are usually vastly different:

$$\dim M_g = 3g - 3 \quad \text{and} \quad \dim A_g = \frac{g(g+1)}{2}.$$

The characterization of M_g as a sublocus of A_g is called the *Shottky problem*.

§5.1. **Genus 2 curves: analysis of the canonical ring.** Let's start with a basic Riemann–Roch analysis of a genus 2 curve C. We fix a canonical divisor K. We have

$$\deg K = 2 \times g - 2 = 2 \quad \text{and} \quad l(K) = g = 2.$$

So we can assume that

$$K \geq 0$$

is an effective divisor. by Riemann–Roch, for any point $P \in C$,

$$l(K-P) - l(K-(K-P)) = 1 - 2 + \deg(K-P) = 0.$$

Since $l(P) = 1$ (otherwise C is isomorphic to $\mathbb{P}^1$), we have $l(K-P) = 1$. So $|K|$ has no fixed part, and therefore gives a degree 2 map

$$\phi_{|K|} : C \to \mathbb{P}^1.$$

By Riemann–Hurwitz, it has 6 ramification points called *Weierstrass points*. We also see that C admits an involution permuting two branches of $\phi_{|2K|}$. It is called the *hyperelliptic involution*.

Now consider $|3K|$. By Riemann–Roch, we have $l(3K) = 5$ and $l(3K - P - Q) = 3$ for any points $P, Q \in C$. It follows that $|3K|$ is very ample and gives an embedding

$$C \hookrightarrow \mathbb{P}^4.$$

To get a bit more, we observe that most of geometry of C is nicely encoded in the *canonical ring*

$$R(K) = \bigoplus_{n=0}^{\infty} \mathcal{L}(nK).$$

We can give a more general definition:

5.1.1. DEFINITION. Let $D \geq 0$ be an effective divisor on a curve C. Its graded algebra is defined as follows:

$$R(D) = \bigoplus_{n=0}^{\infty} \mathcal{L}(nD).$$

This is a graded algebra: notice that if $f \in \mathcal{L}(aD)$ and $g \in \mathcal{L}(bD)$ then

$$(fg) + (a+b)D = (f) + aD + (g) + bD \geq 0,$$

so $fg \in \mathcal{L}(a+b)D$.

5.1.2. REMARK. We have only defined divisors on curves in this class, but in principle it is no harder to defined a graded algebra of any divisor on an algebraic variety of any dimension. The canonical ring $R(K)$ of a smooth variety of dimension n was a subject of a really exciting research in the last 30 years which culminated in the proof of a very important theorem of Siu and Birkar–Cascini–Hacon–McKernan: $R(K)$ is a finitely generated algebra. This does not sound like much, but it allows us to define $\operatorname{Proj} R(K)$, the so-called canonical model of X. It is easy to see that it depends only on the field of rational functions $\mathbb{C}(X)$. In the curve case, C is uniquely determined by its field of functions, by in dimension > 1 it is easy to modify a variety without changing its field of rational functions (e.g. by blow-ups). So it is very handy to have this canonical model of the field of rational functions. There exists a sophisticated algorithm, called the Minimal Model Program, which (still conjecturally) allows one to construct the canonical model by performing a sequence of basic "surgeries" on X called divisorial contractions and flips.

We can compute the Hilbert function of $R(K)$ by Riemann–Roch:

$$h_n(R(K)) = l(nK) = \begin{cases} 1 & \text{if} \quad n = 0 \\ 2 & \text{if} \quad n = 1 \\ 3 & \text{if} \quad n = 2 \\ 5 & \text{if} \quad n = 3 \\ 2n-1 & \text{if} \quad n \geq 2. \end{cases}$$

Let's work out the generators. $\mathcal{L}(0) = \mathbb{C}$ is generated by 1. This is a unity in $R(K)$. Let x_1, x_2 be generators of $\mathcal{L}(K)$. One delicate point here is that we can (and will) take x_1 to be $1 \in \mathbb{C}(C)$, but it should not be confused with a previous 1 because it lives in a different degree in $R(K)$! In other words, $R(K)$ contains a graded polynomial subalgebra $\mathbb{C}[x_1]$, where any power x_1^n is equal to 1 as a rational function on C.

Any other element of first degree has pole of order 2 at K (because if it has a pole of order 1, it would give an isomorphism $C \simeq \mathbb{P}^1$.

A subalgebra $S = \mathbb{C}[x_1, x_2]$ of R is also a polynomial subalgebra: if we have some homogeneous relation $f(x_1, x_2)$ of degree d then we have

$$f(x_1, x_2) = \prod_{i=1}^{d} (\alpha_i x_1 + \beta_i x_2) = 0 \quad \text{in} \quad \mathbb{C}(C),$$

which implies that $\alpha_i x_1 + \beta_i x_2 = 0$ for some i, i.e. that x_1 and x_2 are not linearly independent, contradiction.

The Hilbert function of S is

$$h_n(S)=\begin{cases}1 & \text{if} \quad n=0\\ 2 & \text{if} \quad n=1\\ 3 & \text{if} \quad n=2\\ 4 & \text{if} \quad n=3\\ n & \text{if} \quad n\geq 2.\end{cases}$$

So the next generator we need for $R(K)$ is a generator y in degree 3.

What happens in degree 4? We need 7 elements and we have 7 elements

$$x_1^4,\ x_1^3x_2,\ x_1^2x_2^3,\ x_1x_2^3,\ x_2^4,\quad yx_1, yx_2.$$

We claim that they are indeed linearly independent, and in fact we claim:

5.1.3. LEMMA. *There is no linear relation in $\mathbb{C}(C)$ of the form*

$$yf_k(x_1,x_2)=f_{k+3}(x_1,x_2),$$

where the lower index is the degree. In particular, $R(K)$ is generated by x_1, x_2, y.

Proof. Suppose the linear relation of the form above exists. Then y, as a rational function on C, is a rational function $f(x_1,x_2)$. One can show that this is impossible either by an elementary analysis of possible positions of roots of y and this rational function $f(x_1,x_2)$ or by simply invoking the fact that as we already know $3K$ is very ample, and in particular functions in $|3K|$ separate points of C. But if y is a rational function in x_1 and x_2 then y takes the same values on two points from each fiber of $\phi_{|2K|}$. □

It follows that

5.1.4. LEMMA. *$R(K)$ is isomorphic to a polynomial algebra in x_1, x_2, y modulo a relation*

$$y^2=f_6(x_1,x_2),$$

where f_6 is a polynomial of degree 6.

Proof. We already know that $R(K)$ is generated by x_1, x_2, y, and that $y\notin\mathbb{C}(x_1,x_2)$. It follows that y^2, $y\mathbb{C}[x_1,x_2]_3$, and $\mathbb{C}[x_1,x_2]_6$ are linearly dependent in $R(K)_6$ and this gives the only relation in $R(K)$:

$$y^2=yf_3(x_1,x_2)+f_6(x_1,x_2).$$

We can make a change of variables $y'=y-\frac{1}{2}f_3$ to complete the square, which brings the relation in the required form. □

§5.2. **Graded algebra of an ample divisor.** Now let's interpret these algebraic results geometrically. The basic fact is:

5.2.1. LEMMA. *If D is an ample divisor on a curve C then* $\operatorname{Proj} R(D)=C$.

Proof. If D is very ample and $R(D)$ is generated by $R(D)_1$ then $R(D)$ is isomorphic to a a polynomial algebra in $x_0,\ldots,x_N\in\mathcal{L}(D)$ modulo the relations that they satisfy, i.e. $R(D)=\mathbb{C}[x_0,\ldots,x_N]/I$, where I is a homogeneous ideal of $C\subset\mathbb{P}^N$. So in this case clearly $\operatorname{Proj} R(D)=C$. In general, if D is ample then kD is very ample for some $k>0$. Also, we know by Lemma 4.5.3 that the Veronese subalgebra $R(lD)=R(D)^{(l)}$ is generated by its first graded piece for some $l > 0$. So klD is a very ample divisor and $R(klD) = R^{(kl)}$ is generated by its first graded piece. Then we have $\operatorname{Proj} R(D) = \operatorname{Proj} R(klD) = C$. We are not using here that C is a curve, so if you know your divisors in higher dimension, everything works just as nicely. □

As a corollary, we have

5.2.2. COROLLARY. *Let C be a genus 2 curve. Then $R(K)$ induces an embedding*

$$C \subset \mathbb{P}(1,1,3)$$

and the image is defined by an equation

$$y^2 = f_6(x_1, x_2). \tag{5.2.3}$$

The embedding misses a singularity of $\mathbb{P}(1,1,3)$ (where $x_1 = x_2 = 0$, $y = 1$). In the remaining two charts of $\mathbb{P}(1,1,3)$, the curve is given by equations

$$y^2 = f_6(1, x_2) \quad \text{and} \quad y^2 = f_6(x_1, 1).$$

The projection onto $\mathbb{P}^1_{[x_1:x_2]}$ is a bicanonical map $\phi_{|2K|}$ and roots of f_6 are branch points of this $2:1$ cover. In particular, f_6 has no multiple roots and any equation of the form (5.2.3) *defines a genus 2 curve.*

The tricanonical embedding $C \subset \mathbb{P}^4$ factors through the Veronese embedding

$$\mathbb{P}(1,1,3) \hookrightarrow \mathbb{P}^4, \quad (x_1, x_2, x_3, y) \mapsto [x_1^3 : x_1^2 x_2 : x_1 x_2^2 : x_2^3 : y],$$

where the image is a projectivized cone over a rational normal curve.

This sets up a bijection between curves of genus 2 and unordered 6-tuples of distinct points $p_1, \ldots, p_6 \in \mathbb{P}^1$ modulo PGL_2. We are going to use this to construct M_2. The classical way of thinking about 6 unordered points in $\mathbb{P}^1$ is to identify them with roots of a binary form $f_6(x_1, x_2)$ of degree 6. Let V_6 be a vector space of all such forms and let $D \subset \mathbb{P}(V_6)$ be the discriminant hypersurface (which parameterizes binary sextics with multiple roots). Thus we have (set-theoretically):

$$M_2 = (\mathbb{P}(V_6) \setminus D)/\operatorname{PGL}_2 .$$

§5.3. **GIT: Proj quotient.** We will construct the quotient $\mathbb{P}(V_6)/\operatorname{PGL}_2$ and then through away the image of D from it to get M_2. So far we were only taking quotients of affine varieties by the action of the group. How about quotients of projective varieties?

5.3.1. EXAMPLE. Here is a preview: what is the quotient of the standard $\mathbb{P}^2$ by the action of the symmetric group S_3 that acts by permuting the coordinates x_1, x_2, x_3? We can realize $\mathbb{P}^2$ as the quotient of $\mathbb{A}^3$ by the action of $\mathbb{C}^*$, which commutes with the action of S_3. So we can take the quotient by the action of S_3 first, which gives $\mathbb{A}^3$ with coordinates given by the elementary symmetric functions. Now we can quotient out by the action of $\mathbb{C}^*$ but now notice that it has weights $1, 2, 3$! So the quotient morphism is

$$\pi : \mathbb{P}^2 \to \mathbb{P}(1,2,3),$$

$$[x_1 : x_2 : x_3] \mapsto [x_1 + x_2 + x_3 : x_1x_2 + x_2x_3 + x_1x_3 : x_1x_2x_3].$$

More systematically, the procedure is as follows. Suppose a group G acts on a projective variety X. Suppose we can write $X = \operatorname{Proj} R$, where R is some finitely generated graded algebra. This is called a *choice of polarization*. Suppose we can find an action of G on R that induces an action of G on X. This is called a *choice of linearization*. Then we can form a GIT quotient

$$X//G = \operatorname{Proj} R^G.$$

In the example above, $\mathbb{P}^2 = \operatorname{Proj} \mathbb{C}[x_1, x_2, x_3]$, and

$$\mathbb{P}^2//S_3 = \operatorname{Proj} \mathbb{C}[x_1, x_2, x_3]^{S_3} =$$
$$= \operatorname{Proj} \mathbb{C}[x_1 + x_2 + x_3, x_1x_2 + x_2x_3 + x_1x_3, x_1x_2x_3] = \mathbb{P}(1, 2, 3).$$

We will use this construction to describe M_2.

§5.4. **Classical invariant theory of a binary sextic.** We have to describe the algebra $R = \mathcal{O}(V_6)^{\mathrm{SL}_2}$ of SL_2-invariant polynomial functions for the linear action of SL_2 on V_6. The classical convention for normalizing the coefficients of a binary form is to divide coefficients by the binomial coefficients:

$$f_6 = ax^6 + 6bx^5y + 15cx^4y^2 + 20dx^3y^3 + 15ex^2y^4 + 6fxy^5 + gy^6.$$

Explicit generators for R were written down in the 19-th century by Clebsch, Cayley, and Salmon. We are not going to prove that they indeed generate the algebra of invariants but let's discuss them to see how beautiful the answer is. Let $p_1, \ldots, p_6$ denote the roots of the dehomogenized form $f_6(x, 1)$ and write (ij) as a shorthand for $p_i - p_j$. Then we have the following generators (draw some graphs):

$$I_2 = a^2 \sum_{\text{fifteen}} (12)^2(34)^2(56)^2$$

$$I_4 = a^4 \sum_{\text{ten}} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2$$

$$I_6 = a^6 \sum_{\text{sixty}} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2(14)^2(25)^2(36)^2$$

$$D = I_{10} = a^{10} \prod_{i<j} (ij)^2$$

$$I_{15} = a^{15} \sum_{\text{fifteen}} ((14)(36)(52) - (16)(32)(54)).$$

Here the summations are chosen to make the expressions S_6-invariant. In particular, they can all be expressed as polynomials in $\mathbb{C}[a, b, c, d, e, f, g]$, for example

$$I_2 = -240(ag - 6bf + 15ce - 10d^2). \tag{5.4.1}$$

Here is the main theorem:

5.4.2. THEOREM. *The algebra $R = \mathcal{O}(V_6)^{\mathrm{SL}_2}$ is generated by invariants I_2, I_4, I_6, I_{10}, and I_{15}. The subscript is the degree. Here $D = I_{10}$ is the discriminant which vanishes iff the binary form has a multiple root. The unique irreducible relation among the invariants is*

$$I_{15}^2 = G(I_2, I_4, I_6, I_{10}).$$

Now we use our strategy to construct M_2:

- Compute $V_6//\operatorname{SL}_2 = \operatorname{MaxSpec} R$ first. By 19-th century, this is
$$\mathbb{C}[I_2, I_4, I_6, I_{10}, I_{15}]/(I_{15}^2 = G(I_2, I_4, I_6, I_{10})).$$
- Now quotient the result by $\mathbb{C}^*$, i.e. compute $\operatorname{Proj} R$. Here we have a magical simplification: $\operatorname{Proj} R = \operatorname{Proj} R^{(2)}$ but the latter is generated by I_2, I_4, I_6, I_{10}, and I_{15}^2. Since I_{15}^2 is a polynomial in other invariants, in fact we have
$$\operatorname{Proj} R^{(2)} = \operatorname{Proj} \mathbb{C}[I_2, I_4, I_6, I_{10}] = \mathbb{P}(2,4,6,10) = \mathbb{P}(1,2,3,5).$$
- To get M_2, remove a hypersurface $D = 0$, i.e. take the chart $D_{I_{10}}$ of $\mathbb{P}(1,2,3,5)$. This finally gives
$$M_2 = \mathbb{A}^3/\mu_5,$$
where μ_5 acts with weights $1, 2, 3$.
- One can show that $\mathbb{C}[A, B, C]^{\mu_5}$ has 8 generators. So as an affine variety, we have
$$M_2 = (\mathbb{P}(V_6) \setminus D)/\operatorname{PGL}_2 \hookrightarrow \mathbb{A}^8,$$
$$\{y^2 = f(x)\} \mapsto \left(\frac{I_2^5}{I_{10}}, \frac{I_2^3 I_4}{I_{10}}, \frac{I_2 I_4^2}{I_{10}}, \frac{I_4^5}{I_{10}^2}, \frac{I_2^2 I_6}{I_{10}}, \frac{I_2 I_6^3}{I_{10}^2}, \frac{I_6^5}{I_{10}^3}, \frac{I_4 I_6}{I_{10}}\right).$$

This of course leaves more questions then gives answers:

(1) How do we know that points of M_2 correspond to isomorphism classes of genus 2 curves? In other words, why is it true that our quotient morphism
$$\mathbb{P}(V_6) \setminus D \to \mathbb{A}^3/\mu_5$$
is surjective and separates PGL_2-orbits? It is of course very easy to give examples of quotients by infinite group actions that do not separate orbits.

(2) Can one prove the finite generation of the algebra of invariants and separation of orbits by the quotient morphism without actually computing the algebra of invariants?

(3) Is M_2 a coarse moduli space (and what is a family of genus 2 curves)?

(4) Our explicit description of M_2 as $\mathbb{A}^3/\mu_5$ shows that it is singular. Which genus 2 curves contribute to singularities?

(5) Our construction gives not only M_2 but also its compactification by $\operatorname{Proj} R$. Can we describe the boundary $\operatorname{Proj} R \setminus M_2$?

(6) Are there other approaches to the construction of M_2?

§5.5. **Homework due on April 1.**

Write your name here:

Problem 1. Let $C \subset \mathbb{P}^d$ be a rational normal curve of degree d, let $\hat{C} \subset \mathbb{A}^{d+1}$ be the affine cone over it, and let $\bar{C} \subset \mathbb{P}^{d+1}$ be its projective closure. Show that $\bar{C}$ is isomorphic to $\mathbb{P}(1,1,d)$ (1 point).

Problem 2. Let $P \in \mathbb{P}^1$ be a point. Let $R = \bigoplus\limits_{k\geq 0} \mathcal{L}(kP)$ be the associated ring. Describe the projective embedding of $\mathbb{P}^1$ given by the d-th Veronese subalgebra of R (1 point)

Problem 3. Show that I_2 (see (5.4.1)) is indeed an SL_2-invariant polynomial. (2 points)

Problem 4. A weighted projective space $\mathbb{P}(a_0,\ldots,a_n)$ is *well-formed* if no n of the weights $a_0,\ldots,a_n$ have a common factor. For example, $\mathbb{P}(1,1,3)$ is well-formed but $\mathbb{P}(2,2,3)$ is not. Consider the polynomial ring $R = \mathbb{C}[x_0,\ldots,x_n]$, where x_i has weight a_i. (a) Suppose that $d = \gcd(a_0,\ldots,a_n)$. Show that $R^{(d)} = R$ and that $\mathbb{P}(a_0,\ldots,a_n) \simeq \mathbb{P}(a_0/d,\ldots,a_n/d)$. (b) Suppose that $d = \gcd(a_1,\ldots,a_n)$ and that $(a_0,d) = 1$. Compute $R^{(d)}$ and show that $\mathbb{P}(a_0,\ldots,a_n) \simeq \mathbb{P}(a_0, a_1/d\ldots,a_n/d)$. Conclude that any weighted projective space is isomorphic to a well-formed one (2 points).

Problem 5. Compute $\operatorname{Proj}\mathbb{C}[x,y,z]/(x^5+y^3+z^2)$. Here x has weight 12, y has weight 20, and z has weight 30 (1 point).

Problem 6. Using the fact that $M_2 = \mathbb{A}^3/\mu_5$, where μ_5 acts with weights $1,2,3$, construct M_2 as an affine subvariety of $\mathbb{A}^8$ (1 point).

Problem 7. Let V_4 be the space of degree 4 binary forms. Show that $\mathcal{O}(V_4)^{\mathrm{SL}_2}$ is a polynomial algebra generated by invariants of degrees 2 and 3 (hint: use Problem 4 from the previous homework). (3 points).

Problem 8. (a) Prove (4.2.1). (b) Prove (4.4.3) (1 point).

Problem 9. Let $P \in E$ be a point on an elliptic curve. (a) Compute $\operatorname{Proj} R(P)$ and the embedding of E in it. (b) Compute $\operatorname{Proj} R(2P)$ and the embedding of E in it. (2 points)

Problem 10. Let $P \in E$ be a point on an elliptic curve. Show that $\phi_{|4P|}$ embeds E in $\mathbb{P}^3$ as a complete intersection of two quadrics (i.e. the homogeneous ideal of E in this embedding is generated by two quadrics) (2 points).

Problem 11. Show that any genus 2 curve C can be obtained as follows. Start with a line $l \subset \mathbb{P}^3$. Then one can find a quadric surface Q and a cubic surface S containing l such that $Q \cap S = l \cup C$ (2 points).

Problem 12. Assuming that $M_2 = \mathbb{A}^3/\mu_5$ set-theoretically, define families of curves of genus 2 (analogously to families of elliptic curves), and show that M_2 is a coarse moduli space (2 points).

Problem 13. Assuming the previous problem, show that M_2 is not a fine moduli space (2 points).

Problem 14. Show that $\mathbb{A}^n \setminus \{0\}$ is not an affine variety for $n > 1$ (1 point).

Problem 15. Suppose X and Y are separated algebraic varieties. Explain how to define $X \times Y$ as an algebraic variety and show that it is separated (2 points).

Problem 16. (a) Show that an algebraic variety X is separated if and only if the diagonal X is closed in $X \times X$. (b) Show that a topological space X is Hausdorff if and only if the diagonal X is closed in $X \times X$ equipped with a product topology. (c) Explain how (a) and (b) can be both true but $\mathbb{A}^1$ is both separated and not Hausdorff (2 points).

Problem 17. Use affine charts to show that $G(2, n)$ is an algebraic variety without using the Plücker embedding (1 point).

Problem 18. Consider rays $R_1, \ldots, R_k \subset \mathbb{R}^2$ emanating from the origin, having rational slopes, going in the counter-clockwise direction, and spanning the angle 2π once. Suppose that each angle $R_i R_{i+1}$ (and $R_k R_1$) is less than π. This is called a (two-dimensional) *fan*. The angles $R_i R_{i+1}$ (and $R_k R_1$) are called (top-dimensional) cones of the fan. Rays themselves are also (one-dimenesional) cones. The origin is a zero-dimensional cone. Now for each cone σ of the fan, consider the semigroup $\Lambda = \sigma \cap \mathbb{Z}^2$ and the dual semigroup

$$\Lambda^\perp = \{(u, v) \in \mathbb{Z}^2 \quad | \quad ui + vj \geq 0 \quad \text{for any} \quad (i, j) \in \Lambda\} \subset \mathbb{Z}^2.$$

Let K be the field $\mathbb{C}(x, y)$. We can think about an element $(i, j) \in \mathbb{Z}^2$ as a Laurent monomial $x^i y^j$. This gives us algebras $\mathbb{C}[\sigma] \subset K$ spanned by monomials in $\Lambda^\perp$. (a) Show that for each inclusion of cones $\tau \subset \sigma$, $\operatorname{MaxSpec} \mathbb{C}[\tau]$ is a principal open subset in $\operatorname{MaxSpec} \mathbb{C}[\sigma]$. (b) Show that one can glue all $\operatorname{MaxSpec} \mathbb{C}[\sigma]$ together. This is called a *toric surface*. (c) Show that weighted projective planes are toric surfaces (3 points).

Problem 19. An algebraic curve is called bielliptic if it admits a $2 : 1$ morphism $C \to E$ onto an elliptic curve; the covering transformation is called a bielliptic involution. Let C be a genus 2 curve. (a) Show that if C is bielliptic then its bielliptic involution commutes with its hyperelliptic involution. (b) Show that C is bielliptic if and only if the branch locus $p_1, \ldots, p_6 \in \mathbb{P}^1$ of its bi-canonical map has the following property: there exists a $2 : 1$ morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that $f(p_1) = f(p_2)$, $f(p_3) = f(p_4)$, and $f(p_5) = f(p_6)$. (c) Show that (b) is equivalent to the following: if we realize $\mathbb{P}^1$ as a conic in $\mathbb{P}^1$ then lines $\overline{p_1 p_2}$, $\overline{p_3 p_4}$, and $\overline{p_5 p_6}$ all pass through a point (3 points).