STAT 515 - Statistics I

Midterm Exam Two

Name: ____
Section: ____

Instruction:

- Notes, calculators, and other electronics are not allowed.
- Show all your work. You need to fully justify all of your answers. A correct answer with insufficient justification may receive no credit.
- A formula sheet is provided in the last page.
- You have 2 hours to complete the exam. There are 6 problems in the exam.

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Good Luck !

Problem	Points
1	/14
2	/16
3	/22
4	/22
5	/14
6	/12
Total	/100

1. Answer each of these questions with complete justification.

   (a) (5 point) Suppose Y is uniformly distributed over the interval $[0, 2]$. Compute $\mathbf{E}(3Y^3)$.

   (b) (4 points) Suppose Y is normally distributed with mean 1 and variance 2. Set up but do not evaluate an integral that corresponds to $\mathbf{P}(3 < Y)$.

   (c) (5 points) Decide if the each of the following functions define a probability density function (provide full justification):

$$f(x) = \begin{cases} 6(x - x^2) & \text{if} \quad 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}, \quad g(x) = \begin{cases} \frac{1}{3}(x - 1) & \text{if} \quad 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}.$$

2. The lifetime of a certain electric machine is modeled by an exponential distribution with mean β years. Let the random variable Y be the lifetime of the machine.
   (a) (8 points) Find the smallest value of β for which the probability that the product will last at least 4 years is greater than or equal to 0.6 (i.e., such that $\mathbf{P}(Y \geq 4) \geq 0.6$). You can leave your final answer in terms of logarithms.

   (b) (8 points) Now assume that the mean of Y is 5 (that is, $\beta = 5$). If Y is the lifetime of the machine, then the machine will generate a total profit of $10000e^{-2Y+3}$ dollars. Compute the expected profit over the lifetime of the machine. You should compute all integrals but you can leave your final answer in terms of exponentials.

3. Suppose the continuous random variable Y has a probability density function defined by

$$f(y) = \begin{cases} 2e^{2+2y} & \text{if} \quad y \leq -1 \\ 0 & \text{if} \quad y > -1 \end{cases}.$$

(a) (8 points) Compute the cumulative distribution function $F(y)$.

(b) (8 points) Find the moment generating function of Y, that is $\mathbf{E}(e^{tY})$, for $t > -2$.

(c) (6 points) Find the mean of Y. You may use the result from the previous part to compute this (otherwise you can use integration by parts to compute it directly, but this may be more complicated).

4. Suppose the discrete joint density function of the random variables Y_1 and Y_2 is given by

$$\begin{array}{lll} p(-1,-1) = \dfrac{1}{10} & p(-1,0) = \dfrac{2}{10} & p(-1,1) = 0 \\ p(0,-1) = 0 & p(0,0) = \dfrac{3}{10} & p(0,1) = \dfrac{1}{10} \\ p(1,-1) = 0 & p(1,0) = \dfrac{1}{10} & p(1,1) = \dfrac{2}{10}. \end{array}$$

(a) (7 points) Find the marginal probabilities of Y_1 and Y_2. Use a table format if convenient.

(b) (10 points) Compute the probabilities that $Y_2 = y$, for $y = -1, 0, 1$, given that $Y_1 = -1$, and use them to compute the conditional expectation $\mathbf{E}(Y_2 | Y_1 = -1)$.

(c) (5 points) Are Y_1 and Y_2 independent? Justify your answer.

5. Suppose the joint density function of the random variables Y_1 and Y_2 is given by

$$f(y_1, y_2) = \begin{cases} \frac{1}{2} & \text{if } \; 0 \leq y_1 \leq y_2 < 2 \\ 0 & \text{otherwise} \end{cases}.$$

(a) (7 points) Find the marginal density functions of Y_1 and Y_2.

(b) (7 points) Find the conditional distribution function of Y_1 given $Y_2 = 1$ and use it to compute $\mathbf{P}(\frac{1}{2} \leq Y_1 | Y_2 = 1)$.

6. (a) (5 points) Suppose Y_1 and Y_2 are random variables with $\mathbf{E}(Y_1) = 1$, $\mathbf{E}(Y_2) = 2$, $\mathbf{Var}(Y_1) = 1$, $\mathbf{Var}(Y_2) = 4$, and $\mathbf{Cov}(Y_1, Y_2) = 2$. Compute the mean and variance of $3Y_1 - 5Y_2$.

(b) (7 points) Suppose the joint distribution function of Y_1 and Y_2 is given by

$$f(y_1, y_2) = \begin{cases} 6y_2 & 0 \leq y_2 \leq y_1 \leq 1 \\ 0 & \text{othwewise} \end{cases}.$$

One can show that $\mathbf{E}(Y_1) = \frac{1}{2}$ and that $\mathbf{E}(Y_2) = \frac{3}{4}$ (you may assume this). Compute $\mathbf{Cov}(Y_1, Y_2)$. Are Y_1 and Y_2 independent (justify your answer)?

Forumla Sheet

The following formulas may be useful:

- Uniform $[a, b]$: $f(y) = \frac{1}{b-a}$ if $a \leq y \leq b$ and $f(y) = 0$ otherwise. $\mu = \frac{1}{2}(a+b)$, $\sigma^2 = \frac{1}{12}(b-a)^2$.
- Normal with mean μ and variance σ^2: $f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(y-\mu)^2}{2\sigma^2}}$.
- Gamma with mean $\alpha\beta$ and variance $\alpha\beta^2$: $f(y) = \frac{y^{\alpha-1}e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)}$ if $y \geq 0$ and $f(y) = 0$ if $y < 0$, where $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}dy$.
- Exponential with mean β and variance β^2: $f(y) = \frac{1}{\beta}e^{-y/\beta}$ if $y \geq 0$ and $f(y) = 0$ if $y < 0$.
- Beta with mean $\frac{\alpha}{\alpha+\beta}$ and variance $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$: $f(y) = \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)}$ otherwise. Here $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.
- Let $U = \sum_{i=1}^n a_i Y_i$ and $W = \sum_{j=1}^m b_j X_j$ where a_i and b_j are constants and Y_i and X_j random variables. Then $\mathbf{Var}(U) = \sum_{i=1}^n a_i^2 \mathbf{Var}(Y_i) + 2\sum_{i<k} a_i a_k \mathbf{Cov}(Y_i, Y_k)$ and $\mathbf{Cov}(U, W) = \sum_{i=1}^n \sum_{j=1}^m a_i b_j \mathbf{Cov}(Y_i, X_j)$.