University of Massachusetts, Amherst, Department of Mathematics and Statistics

STAT 515 - Statistics I

Final Exam, May 8 2017

Name: ____
Section: ____

Instruction:

- Notes, calculators, and other electronics are not allowed.
- Show all of your work. You need to fully justify all of your answers. A correct answer with insufficient justification may receive no credit.
- A formula sheet is provided on the last page.
- You have 2 hours to complete the exam. There are 6 problems in the exam.

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Good Luck !

Problem	Points
1	/16
2	/16
3	/20
4	/16
5	/17
6	/15
Total	/100

1. Use the central limit theorem to give approximate answers to the following questions. The final answer can be left as an integral.

   (a) (8 point) A supermarket manager asks 100 customers if they are satisfied with a certain product. If the true fraction of customers who are satisfied is 0.8, what is the approximate probability that the sample fraction will be within 0.15 units of the true fraction (that is, within 0.15 of 0.8).

   (b) (8 points) 20% of computer chips from a certain manufacturer are defective. If 100 computer chips are purchased, what is the approximate probability that at least 20 of them are defective?

2. The lifetime of light bulbs from a certain factory has an exponential distribution with mean 18 months.

   (a) (8 points) 81 light bulbs are purchased from this company. Use the central limit theorem to approximate the probability that the average lifetime of the 81 light bulbs is between 14 and 16 months. You can leave the final answer in terms of an integral.

   (b) (8 points) Now suppose we have 100 light bulbs. Use the central limit theorem to approximate the probability that the *total* lifetime of the 100 bulbs (that is, the sum of their lifetimes) does not exceed 2000 months. You can leave the final answer as an integral.

3. (a) (10 points) Suppose the probability density function of Y is

$$f_Y(y) = \begin{cases} \frac{2}{3}(1+2y^3), & \text{if } 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}.$$

If $U = \frac{1}{2}(1-Y)$ find the probability density function, $f_U(u)$, of U. Make sure to specify the domain of f_U.

(b) (10 point) Suppose Y has a Gamma distribution with $\alpha = 1$ and some β. Express the distribution of $U = \frac{2}{\beta}Y$ as one of the distributions on the formula sheet. Make sure to specify all parameters. Hint: You can use moment generating functions.

4. Suppose Y_1 and Y_2 have the joint density function

$$f(y_1, y_2) = \begin{cases} \frac{1}{2}, & 0 \leq y_1 \leq 2,\ 0 \leq y_2 \leq 1 \\ 0, & \text{otherwise} \end{cases} .$$

(a) (10 points) Compute the marginal density functions of Y_1 and Y_2.

(b) (6 points) Are Y_1 and Y_2 independent? Justify your answer.

5. Suppose the density function of the random variable Y is given by

$$f(y) = \begin{cases} \frac{3}{8}y^2 & \text{if} \quad 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}.$$

(a) (9 points) Find the cumulative probability function F of Y.

(b) (8 points) Compute $\mathbf{E}(2Y^3 + 1)$.

6. (a) (8 points) A test for a disease correctly detects the disease in 80% of individuals who have the disease. If a person does not have the disease, the test will report that the individual does not have the disease with probability 70%. Only 2% of the population actually has the disease. A person is chosen randomly from the population and the test indicates that the person has the disease. What is the conditional probability that the person actually has the disease? You can leave your answer in the form of an unsimplified fraction.

   (b) (7 points) The telephone lines at an office are busy (and not answered) about 60% of the time. If you call the office on 4 different occasions, what is the probability that your call will be answered *exactly* once. You can leave your answer in combinatorial form.

Forumla Sheet

The following formulas may be useful:

- $P_k^n = \dfrac{n!}{(n-k)!}$, $\quad C_k^n = \binom{n}{k} = \frac{n!}{k!(n-k)!}$.
- Binomial: $p(y) = \binom{n}{y} p^y (1-p)^{n-y}$, $\mu = np$, $\sigma^2 = np(1-p)$, $m(t) = (pe^t + (1-p))^n$.
- Geometric: $p(y) = p(1-p)^{y-1}$, $\mu = \frac{1}{p}$, $\sigma^2 = \frac{1-p}{p^2}$, $m(t) = \frac{pe^t}{1-(1-p)e^t}$.
- Negative binomial: $p(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r}$, $\mu = \frac{r}{p}$, $\sigma^2 = \frac{r(1-p)}{p^2}$.
- Hypergeometric: $p(y) = \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}}$, $\mu = \frac{nr}{N}$, $\sigma^2 = n\left(\frac{r}{N}\right)\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right)$.
- Poisson: $p(y) = \frac{\lambda^y}{y!} e^{-\lambda}$, $\mu = \lambda$, $\sigma^2 = \lambda$. Moment generating function $m(t) = e^{\lambda(e^t - 1)}$.
- Uniform $[a,b]$: $f(y) = \frac{1}{b-a}$ if $a \le y \le b$ and $f(y) = 0$ otherwise. $\mu = \frac{1}{2}(a+b)$, $\sigma^2 = \frac{1}{12}(b-a)^2$, $m(t) = \frac{e^{bt} - e^{at}}{(b-a)t}$.
- Normal with mean μ and variance σ^2: $f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(y-\mu)^2}{2\sigma^2}}$, $m(t) = e^{\mu t + \frac{t^2\sigma^2}{2}}$
- Gamma with mean $\alpha\beta$ and variance $\alpha\beta^2$: $f(y) = \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)}$ if $y \ge 0$ and $f(y) = 0$ if $y < 0$, where $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy$, $m(t) = (1-\beta t)^{-\alpha}$.
- Chi-square (χ^2) with ν degrees of freedom: $f(y) = \frac{y^{\frac{\nu}{2}-1} e^{-y/2}}{2^{\nu/2}\Gamma(\nu/2)}$, if $y > 0$ and $f(y) = 0$ otherwise, $\mu = \nu$, $\sigma^2 = 2\nu$, $m(t) = (1-2t)^{-\nu/2}$.
- Exponential with mean β and variance β^2: $f(y) = \frac{1}{\beta} e^{-y/\beta}$ if $y \ge 0$ and $f(y) = 0$ if $y < 0$.
- Beta with mean $\frac{\alpha}{\alpha+\beta}$ and variance $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$: $f(y) = \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)}$ otherwise. Here $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.
- Let $U = \sum_{i=1}^n a_i Y_i$ and $W = \sum_{j=1}^m b_j X_j$ where a_i and b_j are constants and Y_i and X_j random variables. Then $\mathbf{Var}(U) = \sum_{i=1}^n a_i^2 \mathbf{Var}(Y_i) + 2\sum_{i<k} a_i a_k \mathbf{Cov}(Y_i, Y_k)$ and $\mathbf{Cov}(U,W) = \sum_{i=1}^n \sum_{j=1}^m a_i b_j \mathbf{Cov}(Y_i, X_j)$.
- If $U = h(Y)$ and h is either increasing or decreasing then $f_U(u) = f_Y(h^{-1}(u)) \left| \frac{dh^{-1}(u)}{du} \right|$.