Name: ______________________________

Section: ______________________________

Stat 515
Midterm Examination II
April 4, 2016
(7:00 p.m. - 9:00 p.m.)

Instructions:

- The total score is 120 points.
- There are 10 questions. Please circle 8 problems below that you want to be graded. Otherwise I will grade the first eight problems. Each problem is worth 15 points. If you do extra ones, each of them is worth 3 points.

  1 2 3 4 5 6 7 8 9 10

- Show **ALL** your work!!
- Some questions have more than one parts. Check carefully to ensure that you don't miss any parts.
- Don't scratch on the line marked **Score** on the bottom of each page.
- The last page is a formula sheet.
- The exam is 2 hours. Good Luck !

1. Customers arrive at a checkout counter in a department store according to a Poisson distribution at an average of seven per hour. Let Y be the number of customers arriving per hour.

(5points) (a) What are the expected value of Y, $E(Y)$ and the variance of Y, $V(Y)$?

[Sol] Since Y has a Poisson distribution at an average of seven per hour, $E(Y) = \lambda = 7$. Then $Y \sim Poi(7)$.
$E(Y) = 7$ and $V(Y) = 7$.

(5points) (b) Find the probability that at least two customers arrive.

[Sol] $P(Y \geq 2) = 1 - P(Y < 2) = 1 - P(Y \leq 1) = 1 - e^{-7}7^0/0! - e^{-7}7^1/1!$

(5points) (c) Is the following statement true or false?

The moment generating function(M.G.F) of Y *is* $m(t) = e^{7(e^t-1)}$.

If you think that this is not true, please provide the correct M.G.F. of Y.

2. (4.169 from textbook) This exercise demonstrates that, in general, the results provided by Tchebysheffs theorem cannot be improved upon. Let Y be a random variable such that

$$p(1) = \frac{1}{18}, \quad p(0) = \frac{16}{18}, \quad p(1) = \frac{1}{18}.$$

(5points) (a) Show that $E(Y) = 0$ and $V(Y) = 1/9$.

(10points) (b) Use the probability distribution of Y to calculate $P(|Y - \mu| \geq 3\sigma)$. Compare this exact probability with the upper bound provided by Tchebysheffs theorem to see that the bound provided by Tchebysheffs theorem is actually attained when $k = 3$.

 Score:____________

3. (4.136 from text book) Suppose that the waiting time for the first customer to enter a retail shop after 9:00 A.M. is a random variable Y with an exponential density function given by

$$f(y) = \begin{cases} \frac{1}{\theta} e^{-y/\theta} & y > 0, \\ 0 & \text{elsewhere,} \end{cases}$$

(5points) (a) Find the moment-generating function for of Y.

(5points) (b) Use the answer from part (a) to find E(Y) .

(5points) (c) Use the answer from part (a) to find V(Y) .

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4. (4.8 from text book)Suppose that Y has the density function

$$f(y) = \begin{cases} ky(1-y) & 0 \leq y \leq 1, \\ 0 & \text{elsewhere} \end{cases}$$

(5points) (a) Find the value of k that makes $f(y)$ a probability density function.

(5points) (b) Find $P(Y \leq .4 | Y \leq .8)$.

(5points) (c) Find the .95-quantile, i.e., find a point $\phi_{.95}$ such that $P(Y \leq \phi_{.95}) = .95$.

Score:____________

5. (4.32 from text book) Weekly CPU time used by an accounting firm has probability density function (measured in hours) given by

$$f(y) = \begin{cases} (3/64)y^2(4-y) & 0 \le y \le 4, \\ 0 & \text{elsewhere} \end{cases}$$

(8points) (a) Find the expected value and variance of weekly CPU time.

(7points) (b) The CPU time costs the firm 200 per hour. Find the expected value and variance of the weekly cost for CPU time.

6. Answer the following questions.

(5points) (a) The operator of a pumping station has observed that demand for water during early afternoon hours has an exponential distribution with mean 100 cfs(cubic feet per second). Find the probability that the demand will exceed 200 cfs during the early afternoon on a randomly selected day.

[Sol] $P(Y > 200) = \int_{200}^{\infty} \frac{1}{100} e^{-y/100} dy = e^{-2} = 0.1353$

(10points) (b) A candy maker produces mints that have a label weight of 20.4 grams. Assume that the weights of these mints have a normal distribution with mean 21.37 and variance 0.16 (i.e., the weights of these mints $\sim N(21.37, 0.16)$). Let Y denote the weight of a single mint selected at random from the production line.

(i) What is the probability that Y is larger or equal to 22.07?

[Sol] Since $Y \sim N(21.37, 0.16)$, $P(Y > 22.07) = P(Z > \frac{22.07-21.37}{0.4}) = P(Z > 1.75) = 0.0401$.

(ii) What is the appropriate value for C such that a randomly chosen single mint has a weight less than C with probability 0.8531?

[Sol] $P(Y < C) = P(Z < \frac{C-21.37}{0.4}) = 0.8531$.

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7. If Y has a probability density function given by

$$f(y) = \begin{cases} 4y^2 e^{-2y} & y > 0, \\ 0 & \text{elsewhere} \end{cases}$$

(5points) (a) State the name of the random variable Y.

(10points) (b) Find $E(Y)$ and $V(Y)$.

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8. Let Y_1 and Y_2 have the joint probability density function given by

$$f(y_1, y_2) = \begin{cases} ky_1y_2 & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(5points) (a) Find the value of k that makes $f(y_1, y_2)$ a joint probability density function.

[Sol] $1 = \int_0^1 \int_0^1 ky_1y_2dy_1dy_2 = k\int_0^1 \int_0^1 y_1y_2dy_1dy_2 = k/4$. Therefore, $k = 4$ The change in depth of a river from one day to the next, measured (in feet) at a specific location, is a random variable Y with the following density function:

$$f(y) = \begin{cases} k & -3 \leq y \leq 3, \\ 0 & \text{otherwise.} \end{cases}$$

Determine the value of k.

(5points) (b) Find $F(0.2, 0.4)$.

[Sol] $F(0.2, 0.4) = P(Y_1 \leq 0.2, Y_2 \leq 0.4) = \int_0^{0.2} \int_0^{0.4} 4y_1y_2dy_1dy_2 = 0.0064$.

(5points) (c) Find $P(0.1 \leq Y_1 \leq 0.3, 0 \leq Y_2 \leq 0.2)$.

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9. Answer the following questions.

(5points) (a) Is the following statement is *true* or *false*? (No justification required)

Suppose Y is a continuous random variable. Then $P(a < Y \leq b) = P(a \leq Y < b)$ where a and b are real numbers with $a < b$.

[Sol] True

(5points) (b) Can the following serve as a cumulative distribution function (C.D.F.) of a random variable Y? Say why or why not(**Justification required**).

$$F(y) = \begin{cases} y & 0 \leq y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

[Sol] False. If $F(y)$ is a cdf, $F(-\infty) = 0$, $F(\infty) = 1$ and $F(y)$ should be nondecreasing.

(5points) (c) If the probability density function (p.d.f.) of a continuous random variable Y is

$$f(y) = \begin{cases} 1 & 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

then $P(Y = \frac{1}{2})$ is _____(fill in the blank).(No justification required)

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10. Gasoline is to be stocked in a bulk tank once at the beginning of each week and then sold to individual customers. Let Y_1 denote the proportion of the capacity of the bulk tank that is available after the tank is stocked at the beginning of the week. Because of the limited supplies, Y_1 varies from week to week. Let Y_2 denote the proportion of the capacity of the bulk tank that is sold during the week. Because Y_1 and Y_2 are both proportions, both variables take on values between 0 and 1. Further, the amount sold, y_2 , cannot exceed the amount available, y_1. Suppose that the joint density function for Y_1 and Y_2 is given by

$$f(y_1, y_2) = \begin{cases} 3y_1 & 0 \leq y_2 \leq y_1 \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Find the probability that less than one-half of the tank will be stocked and more than one-quarter of the tank will be sold. (15 points).

Formula Sheet.

1. Y Poisson distribution (λ): probability function

$$p(y) = \frac{\lambda^y}{y!}e^{-\lambda}, \quad , y = 0, 1, \ldots, \lambda > 0$$

moment generating function

$$m(t) = e^{\lambda(e^t - 1)}$$

$$E(Y) = V(Y) = \lambda.$$

2. Tchebysheffs' Theorem: Let Y be a random variable with finite mean μ and variance σ^2 . Then, for any $k > 0$,

$$P(|Y - \mu| < k\sigma) \geq 1 - \frac{1}{k^2} \qquad \text{or} \qquad P(|Y - \mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

3. Expectation and Variance: if Y is a discrete random variable with pmf $p(y)$

$$E[Y] = \sum_y yp(y), \qquad V(Y) = E(Y - E[Y])^2.$$

If Y is a continuous random variable with probability density function $f(y)$

$$E[Y] = \int_y yp(y)dy \qquad V(Y) = E(Y - E[Y])^2.$$

The Moment Generating Function $m(t)$ for a continuous random variable Y with p.d.f. $f(y)$ is

$$m(t) = E[e^{tY}] = \int_y e^{ty}p(y)dy.$$

4. If Y is a continuous random variable with probability density function $f(y)$ and distribution function $F(y)$, then $\frac{dF(y)}{dy} = f(y)$ and $F(y) = \int_{-\infty}^{y} f(x)dx$.

5. Conditional probability of the event A given B, $P(A|B) = \frac{P(A \cap B)}{P(B)}$, where $P(B) > 0$.

6. Y follows the Gamma distribution (α, β): probability density function

$$f(y) = \frac{y^{\alpha-1}e^{-\frac{y}{\beta}}}{\beta^\alpha \Gamma(\beta)}, \quad y > 0,$$

MGF

$$m(t) = (1 - \beta t)^{-\alpha}$$

$$E(Y) = \alpha\beta \qquad \text{and} \qquad V(Y) = \alpha\beta^2.$$

7. Y follows exponential distribution (β) if Y follows Gamma distribution ($\alpha = 1, \beta$).

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NORMAL CUMULATIVE DISTRIBUTION FUNCTION

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Score:____________

scratch paper

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