Name: ____________________

Section: ____________________

Stat 515
Midterm Examination II
April 4, 2016
(7:00 p.m. - 9:00 p.m.)

Instructions:

- The total score is 120 points.
- There are 10 questions. Please circle 8 problems below that you want to be graded. Otherwise I will grade the first eight problems. Each problem is worth 15 points. If you do extra ones, each of them is worth 3 points.

  1 2 3 4 5 6 7 8 9 10

- Show **ALL** your work!!
- Some questions have more than one parts. Check carefully to ensure that you don't miss any parts.
- Don't scratch on the line marked **Score** on the bottom of each page.
- The last page is a formula sheet.
- The exam is 2 hours. Good Luck !

1. Five identical bowls are labeled 1, 2, 3, 4, and 5. Bowl i contains i white and $5 - i$ black balls, with $i = 1 \sim 5$. A bowl is randomly selected and two balls are randomly selected without replacement from the contents of the bowl.

(5points) (a) What is the probability that both balls selected are white?

(10points) (b) Given both balls selected are white, what is the probability that bowl 3 were selected?

Score:____________

2. Cars arrive at a toll booth according to a Poisson process with mean 80 cars per hour. If the attendant makes a one-minute call, what is the probability that at least 1 car arrives during the call? (15 points)

Score:____________

3. A random variable Y has the density function

$$f(y) = ke^{-y^2/2}, \quad -\infty < y < \infty\,.$$

(5points) (a) Find k

(5points) (b) Find the moment-generating function of Y

(5points) (c) Use the answer from part (b) to find $E[Y]$ and $V[Y]$.

 Score:____________

4. Suppose that a company has determined that the number of jobs per week N varies from week to week and has a Poisson distribution with mean λ. The number of hours to complete each job, Y, is Gamma distributed with parameters α and β. The total time to complete all jobs in a week is $T = \sum_{i=1}^{N} Y_i$.

(5points) (a) What is $E[T \mid N = n]$?

(10points) (b) What is $E[T]$, the expected total time to complete all jobs?

Score:____________

5. Let

$$f(x,y) = \begin{cases} 6xy^2, & x,y \in [0,1], \\ 0, & \text{otherwise.} \end{cases}.$$

(3points) (a) Show that $f(x,y)$ is a probability distribution function.

(3points) (b) Compute $f_X(x)$, the marginal distribution of X, and $f_Y(y)$, the marginal distribution of Y.

(3points) (c) Are X and Y independent? Please support your answer with a proof or calculation.

(3points) (d) Find $E(3X - 4Y)$ and $Var(\sqrt{18}X)$.

(3points) (e) Calculate $Var(X\sqrt{18} - Y\sqrt{80})$.

Score:____________

6. Let Z be a standard normal random variable and let $Y_1 = Z$ and $Y_2 = Z^2$.

(5points) (a) What are $E(Y_1)$, $E(Y_2)$?

(5points) (b) What are $E(Y_1 Y_2)$ and $Cov(Y_1, Y_2)$?

(5points) (c) Are Y_1 and Y_2 independent? Why or Why not?

Score:____________

7. Tet Weibull density function is given by

$$f(y) = \begin{cases} \frac{1}{\alpha} m y^{m-1} e^{-y^m/\alpha}, & y > 0, \\ 0, & \text{otherwise.} \end{cases}.$$

where α and m are positive constants. This density function is often used as a model for the lengths of life of physical systems. Suppose Y has the Weibull density just given. Find

(7points) (a) the density function of $U = Y^m$.

(8points) (b) $E(Y^k)$ for any positive integer k.

Score:____________

8. Answer the following questions

(8points) (a) Suppose that Y_1 and Y_2 are independent, standard normal random variables. Find the density function of $U = Y_1^2 + Y_2^2$.

(7points) (b) Let the random variable Y possess a uniform distribution on the interval (0,1). Derive the distribution of the random variable $W = \sqrt{Y}$.

 Score:_____________

9. Answer the following questions.

(5points) (a) The times that a cashier spends processing individual customer's order are independent random variables with mean 2.5 minutes and standard deviation 2 minutes. What is the approximate probability that it will take more than 4 hours to process the orders of 100 people? (**Hint** use the Central Limit Theorem).

(5points) (b) Is the following statement is *true* or *false*? (No justification required)

Suppose two random variables Y_1 and Y_2 have bivariate normal distribution.
Then zero correlation coefficient (i.e., $\rho = 0$) between two random variables Y_1 and Y_2 implies that they are independent.

(5points) (c) Is the following statement is *true* or *false*? (**Justification required**)

Let Y_1 and Y_2 have joint density function

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)} & y_1 > 0, y_2 > 0 \\ 0 & \text{otherwise} \end{cases}$$

Then, Y_1 and Y_2 are independent.

 Score:____________

10. An airline company is considering a new policy of booking as many as 396 persons on an airplane that can seat only 370. Past studies have revealed that only 89% of the booked passengers actually arrive for the flight. The company decided to book 396 persons.

(7points) (a) Formulate the problem in a way that the Central Limit Theorem (CLT) can be applied and justify the use of the CLT.

(8points) (b) Estimate the probability that if the company books 396 persons, then not enough seats will be available.

 Score:____________

Formula Sheet.

Let $Y_1, \ldots, Y_n$ be random variables with $E(Y_i) = \mu_i$. Define $U = \sum_{i=1}^{n} a_i Y_i$ for constants $a_1, \ldots, a_n$. Then the following hold:

a. $E(U) = \sum_{i=1}^{n} a_i \mu_i$.

b. $Var(U) = \sum_{i=1}^{n} a_i^2 Var(Y_i) + 2 \sum_{1 \leq i < j \leq n} a_i a_j Cov(Y_i, Y_j)$.

Let Y_1 have distribution function $F_1(y_1)$, Y_2 have distribution function $F_2(y_2)$, and Y_1 and Y_2 have joint distribution function $F(y_1, y_2)$. Then Y_1 and Y_2 are said to be **independent** if and only if $F(y_1, y_2) = F_1(y_1)F_2(y_2)$ for every pair of real numbers (y_1, y_2).

If Y_1 and Y_2 are random variables with means μ_1 and μ_2, respectively, the **covariance** of Y_1 and Y_2 is

$$Cov(Y_1, Y_2) = E[(Y_1 - \mu_1)(Y_2 - \mu_2)].$$

Transformation Method Let Y have probability density function $f_Y(y)$. If $h(y)$ is either increasing or decreasing for all y such that $f_Y(y) > 0$, then $U = h(Y)$ has density function

$$f_U(u) = f_Y[h^{-1}(u)] \times |\frac{dh^{-1}(u)}{u}|,$$

Central Limit Theorem: Let $Y_1, Y_2, \ldots, Y_n$ be independent and identically distributed random variables with $E(Y_i) = \mu$ and $Var(Y_i) = \sigma^2 < \infty$. Define

$$U_n = \frac{\sum_{i=1}^{n} Y_i - n\mu}{\sigma\sqrt{n}}$$

where $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ Then the distribution function of U_n converges to the standard normal distribution function as $n \to \infty$.

 Score:____________

NORMAL CUMULATIVE DISTRIBUTION FUNCTION

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Score:____________

scratch paper