

In problems that require reasoning, algebraic calculation, or the use of your graphing calculator, it is not sufficient just to write the answers. You must explain how you arrived at your answers and show all your algebraic calculations.

1. Use any method you prefer for the problems below:

(a) Suppose that Z has a standard normal distribution. Find the density function of the random variable $U = Z^2$. **Hints:** You may identify the distribution of U using the tables provided. If V is normally distributed with mean 0 and variance σ^2 (that is $V \sim \mathcal{N}(0, \sigma^2)$), then $\int_{-\infty}^{\infty} f_V(v) dv = 1$.

(b) Let Z have a uniform distribution on $(0, 1)$. Show that $W = -2 \ln(Z)$ has an exponential distribution.

2. (a) Let Y_1, Y_2, \dots, Y_n be independent and identically distributed random variables such that for $0 < p < 1$, $P(Y_i = 1) = p$ and $P(Y_i = 0) = q = 1 - p$ **for** $i = 1, \dots, n$ (Bernoulli random variables).

(i). Find the moment-generating function for the Bernoulli random variable Y_1 .

(ii). Find the moment-generating function for $W = Y_1 + Y_2 + \dots + Y_n$.

(iii). Determine the distribution of W by using the result in (ii).

2. (b) Suppose that Z_1 and Z_2 are independent, standard normal random variables. Find the density function of $U = Z_1^2 + Z_2^2$. (Hint: Compare to Problem 1(a))

3. An anthropologist wishes to estimate the average height of men on the island of Delos. The population standard deviation is assumed to be 2.5 inches.

(a) If the anthropologist has random samples from 100 men, find the probability that the difference between the sample mean and the (unknown) true population mean will not exceed .5 inches. Explain carefully why and how you used the Central Limit Theorem.

(c) Suppose that the anthropologist wants the difference between the sample mean and the population mean to be less than .4 inches, with probability .95. Find the smallest number of samples needed to achieve this objective.

4. An airline knows (from past data) that 5% of the customers who buy tickets on a JFK-BOS flight do not show up. Suppose the airline sells 175 tickets for this flight, knowing that the specific type of plane has only 171 seats. The airline would like to know what is the probability that a seat will be available for every person holding a reservation that shows up at the airport.
- (a) Set up (but do not solve yet!) the problem using a binomial distribution.
 - (b) Use the Central Limit Theorem to calculate the probability that a seat will be available for every person holding a reservation that shows up at the airport.
 - (c) Go back to the formula in (a) and calculate the probability in question. How does it compare to the result in (b)?

Common Probability Distributions, Means, Variances, and Moment-Generating Functions

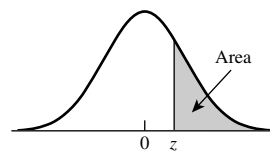
Table 1 Discrete Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Binomial	$p(y) = \binom{n}{y} p^y (1-p)^{n-y};$ $y = 0, 1, \dots, n$	np	$np(1-p)$	$[pe^t + (1-p)]^n$
Geometric	$p(y) = p(1-p)^{y-1};$ $y = 1, 2, \dots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$
Hypergeometric	$p(y) = \frac{\binom{r}{y} \binom{N-r}{n-y}}{\binom{N}{n}};$ $y = 0, 1, \dots, n \text{ if } n \leq r,$ $y = 0, 1, \dots, r \text{ if } n > r$	$\frac{nr}{N}$	$n \left(\frac{r}{N} \right) \left(\frac{N-r}{N} \right) \left(\frac{N-n}{N-1} \right)$	does not exist in closed form
Poisson	$p(y) = \frac{\lambda^y e^{-\lambda}}{y!};$ $y = 0, 1, 2, \dots$	λ	λ	$\exp[\lambda(e^t - 1)]$
Negative binomial	$p(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r};$ $y = r, r+1, \dots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\left[\frac{pe^t}{1-(1-p)e^t} \right]^r$

Table 2 Continuous Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Uniform	$f(y) = \frac{1}{\theta_2 - \theta_1}; \theta_1 \leq y \leq \theta_2$	$\frac{\theta_1 + \theta_2}{2}$	$\frac{(\theta_2 - \theta_1)^2}{12}$	$\frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}$
Normal	$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\left(\frac{1}{2\sigma^2}\right)(y - \mu)^2\right]$ $-\infty < y < +\infty$	μ	σ^2	$\exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$
Exponential	$f(y) = \frac{1}{\beta} e^{-y/\beta}; \beta > 0$ $0 < y < \infty$	β	β^2	$(1 - \beta t)^{-1}$
Gamma	$f(y) = \left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\right] y^{\alpha-1} e^{-y/\beta};$ $0 < y < \infty$	$\alpha\beta$	$\alpha\beta^2$	$(1 - \beta t)^{-\alpha}$
Chi-square	$f(y) = \frac{(y)^{(v/2)-1} e^{-y/2}}{2^{v/2}\Gamma(v/2)};$ $y > 0$	v	$2v$	$(1 - 2t)^{-v/2}$
Beta	$f(y) = \left[\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] y^{\alpha-1}(1 - y)^{\beta-1};$ $0 < y < 1$	$\frac{\alpha}{\alpha + \beta}$	$\frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$	does not exist in closed form

Table 4 Normal Curve Areas
Standard normal probability in right-hand tail
 (for negative values of z , areas are found by symmetry)



z	Second decimal place of z									
	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0722	.0708	.0694	.0681
1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0352	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
2.9	.0019	.0018	.0017	.0017	.0016	.0016	.0015	.0015	.0014	.0014
3.0	.00135									
3.5	.000 233									
4.0	.000 031 7									
4.5	.000 003 40									
5.0	.000 000 287									

From R. E. Walpole, *Introduction to Statistics* (New York: Macmillan, 1968).