Abstract

In this brief talk, we'll try to understand how far are we from answering the question: given a closed $2n$-manifold X and a closed 2-form ω, when is (X, ω) a symplectic manifold? Obvious constrains are that ω^n must induce an orientation and that M must admit a compatible almost complex structure. Amazingly in dimension $2n \geq 6$ these are the only known constraints. In dimension 4, Taubes used the Seiberg-Witten invariants to provide new constrains. Time permitting we will mention how to prove the symplectic Thom conjecture, which is about minimizing the genus of a surface representing a given 2-homology class.

1 The canonical Spin^c-structure

Proposition 1.1 ([4]). *A Spin^c-structure for an oriented vector bundle over X is equivalent (after stabilizing if the fiber dimension is odd or ≤ 2) to a complex structure over the 2-skeleton X^2 that can be extended over the 3-skeleton.*

Proof. We start by observing that the following lifting as a solution (See [4] page 48-49), i.e. there is a map $j\colon BU(n) \to B\mathrm{Spin}^c(2n)$ making the diagram commute.

$$\begin{tikzcd}
 & BS^1 \arrow[d, hook] \\
 & B\mathrm{Spin}^c(2n) \arrow[d] \\
BU(n) \arrow[ur, dashed] \arrow[r, "Bi"] & BSO(2n)
\end{tikzcd}$$

This proves that every time we have an almost complex structure on our vector bundle, there is a Spin^c-structure that comes with it. Sadly there is no uniqueness in general, due to the fact that there is a (unique) obstruction to it in $H^2(BU(n); \pi_2(BS^1)) \cong H^2(BU(n); \mathbb{Z})$.
Let us address the problem of comparing the obstruction for the existence of an almost complex structure and Spin^c-structure for our vector bundle over X. Let us fix some notation: let $F = \mathrm{hofib}(BU(n) \to BSO(2n))$. We start by observing that an almost complex structure always exists on the second skeleton X^2. In fact $\pi_1 BU(n) = 0$ therefore there are no local coefficients to deal with, and $\pi_1 BS^1 = 0$ hence the fiber is k-simple for all k (See Davis and Kirk theorem 7.37) and there is no obstruction for extending to the second skeleton. The extension is not necessarily unique.
Now consider the following commutative cube, given by universal property of the fiber.

$$\begin{tikzcd}
 & F \arrow[rr, dashed, "\exists!\, t"] \arrow[dl] \arrow[dd] & & BS^1 \arrow[dl] \arrow[dd] \\
BU(n) \arrow[rr, "j"] \arrow[dd] & & B\mathrm{Spin}^c(2n) \arrow[dd] & \\
 & * \arrow[rr, equal] \arrow[dl] & & * \arrow[dl] \\
BSO(2n) \arrow[rr, equal] & & BSO(2n) &
\end{tikzcd}$$

We claim that the map $t\colon F \to BS^1$ between the fibers is 3-connected. The proof is a straightforward application of 5-lemma to the l.e.s. in homotopy for the two fibrations and map between them, together with the fact that . Therefore, thanks to the naturality of the obstructions, we can compare the two lifting problems on the lower skeletons at the same time: let $\mathfrak{o}_i \in H^i(X; \pi_{i-1}F)$ be the obstruction to extending the almost complex structure to the i-th skeleton of X. Similarly, let $\mathfrak{h}_i \in H^i(X; \pi_{i-1}BS^1)$ be the obstruction of extending the Spin^c-structure on X^i. By naturality and uniqueness we have that

$t_*\mathfrak{o}_i = \mathfrak{h}_i$, and since the map t is 3-connected, the two extending problems up to the second skeleton are equivalent. In general we observe that the unique obstruction for extending a Spin^c-structure is $\mathfrak{h}_3 \in H^3(X; \pi_2 BS^1) \cong H^3(X; \mathbb{Z})$ and we have $t_*\mathfrak{o}_3 = \mathfrak{h}$. Therefore we conclude that every time we have an almost complex structure extending to the 3rd-skeleton we have a Spin^c-structure on the vector bundle. On the contrary, every time we have a Spin^c-structure on the 2^{nd} skeleton extending to X^3, we know that $0 = \mathfrak{h}_3 \in H^3(X; \pi_2 BS^1)$. But t_* induces an isomorphism $\pi_2 F \to \pi_2 BS^1$, hence

$$0 = \mathfrak{h}_3 = t_*\mathfrak{o}_3$$

Hence $\mathfrak{o}_3 = 0$ □

Lemma 1.2. *In the case of a closed, oriented 4-manifold X, if $\mathfrak{s}$ is the Spin^c-structure arising from an almost complex structure, compatible with the given orientation, then $d(s) = 0$.*

Proof. We first observe that $c_1(\mathfrak{s}) = c_1(\det(\mathbb{S}^+)) = -c_1(T^*X, J)$. This is due to the fact that for a complex bundle $c_1(E) = c_1(\det(E))$. In fact we have

$$\begin{aligned} c_1(\det(\bigwedge^{\text{even}} T^{*0,1}X)) &= c_1(\bigwedge^{\text{even}} T^{*0,1}X) \\ &= c_1(\mathbb{C} \oplus \bigwedge^2 T^{*0,1}X) = c_1(\bigwedge^2 T^{*0,1}X) \\ &= c_1(\det(T^{*0,1}X)) = c_1(T^{*0,1}X) \\ &= c_1(T^*X, -J) = -c_1(T^*X, J) \end{aligned}$$

Since TX is complex,

$$p_1(TX)[X] = (c_1(TX)^2 - 2c_2(TX))[X] = c_1(\mathfrak{s})^2[X] - 2\chi(X)$$

so by Hirzebruch's signature theorem, $c_1(\mathfrak{s})^2[X] - 2\chi(X) = 3\tau(X)$, hence $d(\mathfrak{s}) = 0$. □

During the proof we observed that $c_1(\det(T^*X, J)) = c_1(\bigwedge^2 T^{*0,1}X)$.

Recall that a symplectic manifold, together with a Riemannian metric has a canonical almost complex structure.

Definition 1.3 ([3] remark 3.2)**.** The canonical Spin^c-structure $\mathfrak{s}_0$ is the one for which the positive spinors $\mathbb{S}^+ \cong \Lambda^{0,0} \oplus \Lambda^{0,2}$, i.e. it arises as the canonical polarization of $\Lambda^\bullet T^*X$, thus $c_1(\mathfrak{s}_0) = -c_1(K_X)$, where $K_X = \det(T^*X, J_{\text{can}})$ and $c_1(K_X)$ is the canonical class of the symplectic structure

2 the Seiberg-Witten invariant

(This is taken from [1], page 396, section 10.3). After choosing a Spin^c structure $\mathfrak{s}$ on X, with associated spinor bundles $\mathbb{S}^\pm$ and determinant line bundle $\mathcal{L}$, the object of interest in what follows will be pairs

$$(\varphi, A)$$

where $\varphi \in \Gamma(\mathbb{S}^+)$ is a self-dual spinor field and $A \in \text{Conn}(\mathcal{L})$ is a $U(1)$-connection on $\mathcal{L}$. Namely, we will look at solutions (φ, A) to a couple of mildly-non-linear elliptic partial differential equations and consider such solutions only up to the action of the gauge group $\mathcal{G}(\mathcal{L})$. The Seiberg-Witten equations are:

$$\begin{cases} D_A(\varphi) &= 0 \\ F_A^+ &= \sigma(\varphi) \end{cases}$$

where D_A is the Dirac operator induced by A, while F_A is the imaginary-valued curvature 2-form of A, and $F_A^+ = \frac{1}{2}(F_a + *F_A)$ is its self-dual part. Finally, $\sigma\colon \mathbb{S}^+ \to i\Lambda^2_+$ is the squaring map.

2.1 The moduli space

The solutions (φ, A) to the Seiberg-Witten equations are called (*Seiberg-Witten*) monopoles. The monopoles form a subspace $\mathfrak{S}$ inside the infinite dimensional configuration space $\Gamma(\mathbb{S}^+) \times \mathrm{Conn}(\mathcal{L})$. It's easy to check that the SW equations are invariant under the action of the gauge group $\mathcal{G}(\mathcal{L}) = \{g \colon M \to S^1\}$. Therefore $\mathfrak{S}$ is a gauge-invariant slice of the configuration space. It thus makes sense to factor everything by the action of the gauge group, hence obtaining the moduli space

$$\mathfrak{M} = \mathfrak{S}/\mathcal{G}$$

obviously, this moduli space depends on the choices of Spin^c-structure and Riemannian metric.
If $b_2(X)^+ \geq 1$, then a solution of the SW equations necessarily have $\varphi \neq 0$ for generic choice of Riemannian metric. In this case the action of $\mathcal{G}(\mathcal{L})$ is free, and hence the orbit space has a better chance of being well-behaved. Indeed

1. They are closed orientable manifold
2. If $b_2^+(X) \geq 1$, then for a generic Riemannian metric, the Seiberg-Witten moduli space $\mathfrak{M}$ is either empty or is a smooth manifold of dimension

$$d(\mathfrak{s}) = \frac{1}{4}\left(c_1(\mathfrak{s})^2 - 2\chi(X) - 3\sigma(X)\right)$$

3. If $b_2^+(M) \geq 2$, then, for every two generic metrics g_0 and g_1 and every generic path g_t connecting them, all corresponding moduli spaces $\mathfrak{M}_t$ are smooth manifolds, (possibly empty) and draw a smooth cobordism between $\mathfrak{M}_0$ and $\mathfrak{M}_1$.

Having thus obtained from the Seiberg-Witten equations some very nice moduli spaces, it can be further shown that the natural ambient of $\mathfrak{M}$, the space of all connection-and-spinor pairs modulo gauge-equivalence, has the homotopy type of $CP^\infty \times T^k$ where T^k is the k-dimensional torus. Therefore in the cohomology ring of this ambient here contains a copy of the polynomial ring $\mathbb{Z}[t]$ for a degree 2 class t. Hence if $\mathfrak{M}$ is even-dimensional, then we can evaluate the appropriate class $t^{d/2}$ on it and obtain a numerical invariant of X. Thus we call

$$\mathfrak{sw}_X(\mathfrak{s}) = \langle t^{d/2}, [\mathfrak{M}]\rangle$$

3 Taubes' constraints

In general, on a symplectic manifold (X, ω), the canonical line bundle is the complex line bundle $K_X = \det(T^*X, J)$ where J is a compatible almost complex structure (different J's result in isomorphic line bundles).

Theorem 3.1 (Taubes). *Let (X, ω) be a closed symplectic 4-manifold with $b^+ > 1$. Then:*

- *There is a canonical solution (A_{can}, φ_{can}) to the Dirac equation for $\mathfrak{s}_{can}$. Moreover for $\tau > 0$, $(A_{can}, \tau^{1/2}\varphi_{can})$ (Taubes' monopole) is a solution to the Seiberg Witten equations with 2-form*

$$\eta(\tau) = iF(A^\circ_{can})^+ + \frac{1}{2}\tau\omega$$

- *For τ sufficiently large, the Taubes monopole is the unique solution to $\mathcal{F}_{\eta(\tau)} = 0$ (modulo gauge), and for generic J it is regular. One has*

$$\mathfrak{sw}_{X,\sigma}(0) = 1$$

where $0 = c_1(\mathfrak{s}_0)$ and therefore $\mathfrak{sw}_{X,\sigma}(c_1(K_X)) = (-1)^{1+b_1+b^+}$

- *If $\mathfrak{sw}_{X,\sigma}(c) \neq 0$ then one has*
$$0 \leq c \cdot [\omega] \leq K_X \cdot [\omega]$$
and if one of the inequality is an equality, then either $c = 0$ or $c = K_X$.

We have the following remarkable corollary:

Corollary 3.2. *Let (X, ω) be a closed symplectic 4-manifold with $b^+ > 1$, then $K_X \cdot [\omega] \geq 0$.*

This corollary is the starting point for a successful classification of 4-manifolds admitting symplectic forms with $K_X \cdot [\omega] < 0$ - they have $b^+ = 1$, by the theorem. They turn out to be diffeomorphic to certain complex surfaces, namely blow-ups of $\mathbb{C}P^2, \mathbb{C}P^1 \times \mathbb{C}P^1$, or ruled surfaces.
Taubes constraints are only the simpler part of the picture established by Taubes, in which *SW* invariants are in fact counts of J-holomorphic curves in X. For instance, a deeper reason why $c_1(K_X) \cdot [\omega] \geq 0$: it is that there exists a C^∞-section of K_X whose (transverse) zero-set S - an oriented surface in X - is symplectic: $\omega_{|TS} > 0$.

4 The minimal genus problem

Let X be a closed, orientable 4-manifold, and $\sigma \in H^2(X; \mathbb{Z})$. We know that σ can be represented as the fundamental class $[\Sigma]$ of an oriented, embedded surface $\Sigma \subset X$.

Lemma 4.1. *One can choose Σ to be path connected, and one can make its genus arbitrarily large.*

We want to allow disconnected surfaces $\Sigma = \coprod \Sigma_i$, and to minimize the *complexity*

$$\chi_-(\Sigma) = \sum_{g(\sum_i) > 0} (2g(\Sigma_i) - 2)$$

over representatives Σ of σ (think of the complexity as minus the Euler characteristic)

Definition 4.2. Assume X is oriented with $b^+(X) > 1$. A class $c \in H^2(X; \mathbb{Z})$ is called a **basic** class if it arises as $c_1(\mathfrak{s})$ for a Spin^c-structure $\mathfrak{s}$ such that $\mathfrak{sw}_{X,\sigma}(\mathfrak{s}) \neq 0$

Theorem 4.3. *Assume X is oriented with $b^+(X) > 1$, and suppose σ is represented by an embedded, oriented surface Σ with non-negative normal bundle (that is, each component Σ_i has non-negative self-intersection). Then, for every basic class c, one has*

$$\chi_-(\Sigma) \geq c \cdot \sigma + \sigma \cdot \sigma$$

Let us try to sketch the proof in the case Σ has trivial normal bundle.

Sketch of proof. The starting point is the Gauss-Bonnet formula, which tells us that for a metric h on Σ, one has
$$\int_\Sigma \mathrm{scal}(h) \mathrm{vol}_h = 4\pi\chi(\Sigma)$$
the fact that c is a basic class tells us that, for any metric (and any self-dual 2-form η) the SW equations admit a solution. The idea is to find a suitable metric on the boundary of a tubular nbhd of Σ, which is diffeomorphic to $\Sigma \times S^1 \times [0, 1]$, such that Σ has constant scalar curvature and $\mathrm{vol}_h(\Sigma) = 1$. Now we will use a *stretching* argument somehow common in this kind of proofs. Let X_t be the manifold obtained by *stretching the neck*, namely, replacing $\Sigma \times S^1 \times [0, 1]$ with $\Sigma \times S^1 \times [0, t]$ in the obvious fashion and letting g_t be the new metric. Of course $X_t \cong X_1$. The SW equations on (X_t, g_t) admits solutions for some Spin^c-structure $\mathfrak{s}$ with $c_1(\mathfrak{s}) = c$. Now a standard argument gives the following estimates on the L^2-norm of the curvature F_t associated to a solution for the g_t-SW equations for $\mathfrak{s}$

$$\|F_t\|_{X_t, g_t} \leq 2\pi\chi_-(\Sigma)^2 t^{1/2} + C^{1/2}$$

where C is independent of t. On the other hand, another standard but lengthy argument gives the following lower bound for any closed 2-form on X_t.

$$\|\omega\|_{g_t} \geq t^{1/2} \int_\Sigma \omega$$

now if we apply this relation for $\omega = F_t$, together with the Chern-Weil identity $c_1(TX, J) = \frac{i}{2\pi}[F_t]$ we obtain the following estimate

$$\|F_t\|_{X_t, g_t} \geq 2\pi t^{1/2} \langle c, \sigma \rangle$$

which together with the first one gives

$$\langle c, \sigma \rangle \leq \chi_-(\Sigma) + \frac{C^{1/2}}{2\pi t^{1/2}}$$

by taking the limit for $t \to \infty$ we get the inequality □

In order to prove the result for positive intersections, one blows up several times the surface Σ in order to get a surface with zero self intersection and then use the fact that this operation doesn't change the genus of Σ together with the fact that the Seiberg-Witten invariants are well-behaved when it comes to the blow-up procedure.
A generalization, allowing surfaces of negative self-intersection, is proved by P. Ozsváth and Z. Szabó in [3].

Theorem 4.4. *Suppose that $b^+(X) > 1$ and that σ is represented by an embedded, oriented surface Σ with non-negative normal bundle and without spherical components. If there exists a symplectic form on X, compatible with the orientation, for which $\omega_{|T\Sigma} > 0$, then Σ minimizes χ_- among representatives of σ.*

Proof. One can find an ω-compatible almost complex structure J such that $J(T\Sigma) = T\Sigma$. One then has a short exact sequence of complex vector bundles

$$0 \to T\Sigma \to TX_{|\Sigma} \to N_\Sigma \to 0$$

which implies that for each component Σ_i (with $\sigma_i = [\Sigma_i]$), one has

$$\langle c_1(TX), \sigma_i \rangle = \langle c_1(T\Sigma_i), \sigma_i \rangle + \langle c_1(N_{\Sigma_i}), \sigma_i \rangle$$

which is to say

$$-\chi(\Sigma_i) = \langle c_1(K_X), \sigma_i \rangle + \sigma_i \cdot \sigma_i$$

On the other hand, if $b^+(X) > 1$, the fact that K_X is a basic class (by Taubes) implies an adjunction inequality

$$\chi_-(S) \geq \langle c_1(K_X), \sigma \rangle + \sigma \cdot \sigma$$

in X, valid for any representatives S of σ, symplectic or not. Thus if the symplectic surface Σ has no spherical components, we see that it saturates the adjunction inequality, and so minimizes the complexity (if there are no spherical components, complexity is minus the euler characteristic). □

References

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[3] P. Ozsváth, Z. Szabó *The symplectic Thom conjecture* Annals of Mathematics, **151** (2000), 93-124

[4] R. Gompf *Spin*^{c}*-structures and homotopy equivalences* Geometry and Topology Volume 1 (1997) 41–50.