Abstract

This is a little review of the first part of the paper [1] of Stefan Bauer about refined Seiberg-Witten invariants. All errors are mine and there is no claim of originality in this work.

1 the classical Seiberg-Witten invariant

Let X be a oriented closed 4 manifold. Let $\mathfrak{s}$ be a Spin^c-structure for it. Let $\mathbb{S}^{\pm}$ be the positive and negative Spinor bundles associated to it. Fix a Spin^c-connection A on them, i.e. a unitary connection on the Spinor bundle $\mathbb{S} \cong \mathbb{S}^+ \oplus \mathbb{S}^-$ which is compatible with the Levi-Civita connection on X, in the sense that

$$A_X(Ys) = (\nabla_X^{LC} Y)s + YA_X s$$

for all vector fields X, Y and smooth sections $s \in \Gamma(\mathbb{S})$. Recall the Seiberg-Witten equations can be thought as a fibre preserving S^1-equivariant map between these two S^1-Hilbert bundles over $H^1(X;\mathbb{R})$:

$$\begin{aligned} \widetilde{\mathcal{A}} &= (A + i\ker(\mathrm{d})) \times \big(\Gamma(\mathbb{S}^+) \oplus H^0(X;\mathbb{R}) \oplus \Omega^1(X)\big) \\ \widetilde{\mathcal{C}} &= (A + i\ker(\mathrm{d})) \times \big(\Gamma(\mathbb{S}^-) \oplus \Omega^0(X) \oplus H^1(X;\mathbb{R}) \oplus \Omega^+(X)\big) \end{aligned}$$

The map $\widetilde{\mu}\colon \widetilde{\mathcal{A}} \to \widetilde{\mathcal{C}}$ is defined by

$$(A',\, \varphi,\, f,\, a) \mapsto \big(A',\, D_{A'}\varphi + ia\varphi,\, \mathrm{d}^*a + f,\, a_{\mathrm{harm}},\, \mathrm{d}^+a + \sigma(\varphi)\big)\,. \tag{1}$$

Here $\sigma(\varphi)$ denotes the trace free endomorphism $i(\varphi \otimes \varphi^* - \frac{1}{2}\|\varphi\|^2 \,\mathrm{id})$ of $\mathbb{S}^+$, considered via the map ρ as a selfdual 2-form on X. Restricted to forms, the map is familiar from Hodge theory: It is linear, injective with cokernel the space $H^+(X;\mathbb{R})$ of harmonic selfdual two-forms on X.

The Gauge group $\mathcal{G} = \mathrm{Aut}_{\mathrm{Id}}(\mathfrak{s}) \cong \mathrm{map}(X, S^1)$ acts on spinors on the 4-manifold via multiplication with $u\colon X \to S^1$ and on Spin^c-connections via addition of $ud(u^{-1})$. It acts trivially on forms.

The map $\widetilde{\mu}$ is equivariant with respect to the action of $\mathcal{G}$. Dividing by the free action of the pointed gauge group we obtain the monopole map

$$\mu = \widetilde{\mu}/\mathcal{G}_0 : \mathcal{A} \to \mathcal{C}$$

as a fiber preserving map between the bundles $\mathcal{A} = \widetilde{\mathcal{A}}/\mathcal{G}_0$ and $\mathcal{C} = \widetilde{\mathcal{C}}/\mathcal{G}_0$ over $\mathrm{Pic}^{\mathfrak{s}}(X)$. The preimage of the section $(A',\, 0,\, 0,\, 0,\, -F^+_{A'})$ of $\mathcal{C}$, divided by the residual S^1-action, is called the moduli space of monopoles. The key fact about Seiberg-Witten theory is that the space of solutions with non-zero spinor component (called *irreducibles*) is a closed orientable smooth manifold. For generic choice of perturbations and metric, under the assumption $b^+ > 0$, there are no reducible solution to the SW equations and the irreducible ones are cut out transversely.

Claim 1.1. *The moduli space of irreducible monopoles is a closed orientable manifold, denoted M_μ of dimension $d := ind_{\mathbb{R}}(D) + (b_1 - b^+ - 1)$*

We are almost ready to define the SW-invariant for a manifold X. Let $\mathcal{B}^{\mathrm{irr}}(\mathfrak{s})$ be the quotient via the Gauge action of the irreducible configurations in $\mathcal{A}$. Then a careful inspection reveals that $\mathcal{B}^{\mathrm{irr}}(\mathfrak{s}) \simeq \mathrm{Pic}^{\mathfrak{s}}(X) \times P\Gamma(\mathbb{S}^+)$, hence

$$H^*(\mathcal{B}^{\mathrm{irr}}(\mathfrak{s});\mathbb{Z}) \cong H^*(\mathrm{Pic}^{\mathfrak{s}}(X);\mathbb{Z}) \otimes \mathbb{Z}[t]$$

with $|t| = 2$. Then

$$\mathfrak{sw}_{X,\mathfrak{s}} := \langle t^{d/2}, [M_\mu]\rangle$$

if d is even, otherwise they are set to be 0.

2 Construction of the refined invariants

Before stating the theorem let us fix some notation: let $\mathcal{E}, \mathcal{F}$ be infinite dimensional Hilbert space bundles over a compact base B. The structure group is the orthogonal group with its norm topology. Consider the set of $\mathcal{P}(\mathcal{E}, \mathcal{F})$ of fiber-preserving continuous maps $\sigma\colon \mathcal{E} \to \mathcal{F}$ s.t. they are S^1-equivariant and such that the preimages of bounded sets are bounded. Now let $l\colon \mathcal{E} \to \mathcal{F}$ be a fixed fibrewise Fredholm operator, define $\mathcal{P}_l(\mathcal{E}, \mathcal{F}) \subseteq \mathcal{P}(\mathcal{E}, \mathcal{F})$ to be the subspace consisting of elements φ such that $\varphi - l$ is fibrewise a compact map.

Now let $p\colon \mathcal{F} \cong B \times \mathcal{U} \to \mathcal{U}$ be a trivialisation and suppose $l\colon \mathcal{E} \to \mathcal{F}$ is a continuous fiberwise linear Fredholm map. Then there is a notion of index of a family for Fredholm operators $\operatorname{ind}(l) = E - \underline{U}$, where $E = l^{-1}(\underline{U})$. Think of $\underline{U}$ as representing the coker of l and E the kernel of l. The construction ensure that $\underline{U}$ and E are both finite dimensional subbundles.
Now define

$$\pi^0_{\mathcal{U}}(B; \operatorname{ind} l) := \{\Sigma^{-U} TE, S^0\}_{\mathcal{U}}$$

We have equivariant counterparts for these notions.
Before proving the main theorem, we need to introduce a few technical results:

Lemma 2.1. *Assume $[\varphi] \in \pi^0_{\mathcal{U}}(B; ind\, l)$ admits a representative $\varphi\colon TE \to S^{\underline{U}}$ which is not surjective, then for some choice of trivialization p, $[p\varphi]^+ = 0$ in $\pi^0_{\mathcal{U}}(B; ind\, l)$.*

Proof. By assumption there is a fiber (and hence a projection p) such that $p\varphi\colon TE \to S^U$ is not surjective, therefore it's nullhomotopic. □

Lemma 2.2. *Let $ind\, l = F_0(V) - F_1(V)$. There exist finite dimensional linear G-subspaces $V \subset H$ such that the following hold:*

1. *For every $y \in Y$, the subspace V is mapped onto* coker *$(l_y : E'_y \to H)$. In particular, $F_0(V)$ is a bundle over Y and $\lambda = F_0(V) - F_1(V)$ represents the virtual index bundle* $\operatorname{ind}(l)$.
2. *For any G-linear $W = W' + V$ with $W' \subset V^\perp$, the restricted map $f(W)^+ = (pr_H \circ f)|^+_{F_0(W)}$: $TF_0(W) \to H^+$ misses the unit sphere $S(W^\perp)$.*
3. *The maps $\rho_W f(W)^+$ and $\mathrm{id}_{W'^+} \wedge \rho_V f(V)^+$ are G-homotopic as pointed maps*

$$F_0(W)^+ \cong W'^+ \wedge F_0(V)^+ \to W'^+ \wedge V^+ = W^+. \quad \square$$

Proof. This is Lemma 2.5 in [2] □

Theorem 2.3. *A projection $p\colon \mathcal{F} \cong B \times \mathcal{U} \to \mathcal{U}$ induces a map*

$$\pi_0\left(\mathcal{P}_l(\mathcal{E}, \mathcal{F})\right) \to \pi^0_{\mathcal{U}}(B; ind\, l)$$

Proof. Let's briefly sketch a proof of the theorem: An element $\xi \in \pi^0_{\mathcal{U}}(B; \operatorname{ind} l)$ is represented by a virtual bundle $E - \underline{U}$ over B, together with a map $TE \to S^U$. It may be necessary to suspend the given map in order that it can be replaced by a homotopic map for which the preimage of the base point consists only of the base point. In particular, ξ then is represented by a proper map $E \to \underline{U}$. The given embedding of E into $\mathcal{E}$ and an identification of the orthogonal complements $E^\perp \subset \mathcal{E}$ and $\underline{U}^\perp \subset \mathcal{F}$, results in an element of $\mathcal{P}_l(\mathcal{E}, \mathcal{F})$.
For a given element $\varphi \in \mathcal{P}_l(\mathcal{E}, \mathcal{F})$, choose a real number $R > 0$ and an ϵ with $0 < \epsilon < R$. The boundedness property of φ implies that the preimage under $p\varphi$ of the ball of radius R in $\mathcal{U}$ is bounded in $\mathcal{E}$. Using compactness of B, this bounded preimage is mapped by the fiberwise compact operator $p \circ (\varphi - l)$ into a compact subset of $\mathcal{U}$. We may cover this image with finitely many ϵ-balls, the centers of which generate a finite dimensional vector space $U \subset \mathcal{U}$. After possibly enlarging U, we

can assume that the virtual bundle $E - \underline{U}$ with $E = (pl)^{-1}(U)$ represents $\operatorname{ind} l$. The restriction $p\varphi|_E$ by construction misses the sphere $S_R(U^\perp)$ of radius R in the orthogonal complement of $U \subset \mathcal{U}$. This map $p\varphi|_E$ extends to the one-point completions to give a continuous map

$$TE \to S^{\mathcal{U}} \setminus S_R(U^\perp).$$

Composition with a homotopy inverse to the inclusion $S^U \to S^{\mathcal{U}} \setminus S_R(U^\perp)$ defines an element of $\{T(\operatorname{ind} l), S^0\}$.

It remains of course to be checked that the two constructions lead to well defined maps between the sets in the theorem which are inverse to each other. We will consider two cases: l surjective or not. Assume that $l\colon \mathcal{E} \to \mathcal{F}$ is surjective. Then its index is simply $\operatorname{ind} l = [\ker l]$ since the cokernel is by assumption trivial and therefore the rank of the kernel is constant and they form a vector bundle. By applying lemma 2.2 to our setting, we start with a map $\xi\colon T\ker l \to S^0$ and we consider $\xi + l$ as a result of our construction. Now after applying lemma 2.2 we can always restrict to $[\ker l]$ which gives $[\xi + l]_{|[\ker l]} = [\xi]$ hence proving that the second construction is inverse of the other. Now consider the case where l is not surjective. This means that there is a point on a fiber of $\mathbb{F}$ which is not hit by l. Consider the projection on that fiber, and repeating the same reasoning as before, we apply Lemma 2.1 to the homotopy class $[\xi + l]^+\colon TE \to S^U$ to obtain an homotopy from it to $[\xi]^+$.
To show that the second construction has as right inverse the first one, one has to construct paths in $\mathcal{P}_l(\mathcal{E}, \mathcal{F})$ from an arbitrary element to an element, which can be "projected" onto the image of the first construction. Such a path is made explicit through the following homotopy φ_t which starts from $\varphi = \varphi_0$ and ends at φ_1. It is constant on a disk bundle of radius Q in $\mathcal{E}$, which contains the preimage of an R-disk bundle in $\mathcal{F}$. Outside it is defined by

$$\varphi_t(e) = \left(\frac{|e|}{Q}\right)^t \varphi\left(\left(\frac{|e|}{Q}\right)^{-t} e\right). \tag{2}$$

Notice then every compact map on a bounded set can be approximate uniformly by a finite rank map. The same reason as before (divide in the two cases l surjective and not) applied to the finite rank map proves the fact that this construction has right inverse. $\square$

There is an equivariant version of this theorem which works whenever we have an equivariant projection p.

Remark 2.4. A Fredholm map $\varphi \in \mathcal{P}_l(\mathcal{E}, \mathcal{F})$, for which $p\varphi$ is not surjective, describes the zero element in the stable cohomotopy group associated to its linearization l.

To see this, recall from the previous section that φ is proper. In particular, the image of $p\varphi$ is a closed subset in U (B is compact hence p is closed, then by Theorem 1.4 in [4] a Fredholm map is closed if and only if its proper. If a point $u \in \mathcal{U}$ is in the complement of the image, then so is a whole ϵ-neighbourhood of u. Now apply the construction above for some $R > |u| + \epsilon$ and replace φ by the homotopic φ_1. We choose U so that it also contains u. The map $p\varphi_1|_E$, followed by the orthogonal projection to U is proper and by construction misses u. So its one-point compactification is null homotopic.

Theorem 2.5. *The monopole map $\mu : \mathcal{A} \to \mathcal{C}$ defines an element in the equivariant stable cohomotopy group*

$$\pi^0_{S^1,\mathcal{U}}\left(Pic^{\mathfrak{s}}(X); \operatorname{ind}(D) - \underline{H^+(X;\mathbb{R})}\right).$$

For $b^+ > \dim(Pic^{\mathfrak{s}}(X)) + 1$, a homology orientation determines a homomorphism of this stable cohomotopy group to $\mathbb{Z}$, which maps $[\mu]$ to the integer valued Seiberg-Witten invariant.

We divide the proof in several claims:

Claim 2.6. *The index of the linearization of μ is* $\operatorname{ind}(D) - \underline{H^+(X;\mathbb{R})}$

Proof. it's enough to consider the index of the Fredholm component of the monopole map, since it coincides with the derivative up to zeroth-order operators which don't influence the index. So let us consider the map

$$\begin{aligned} D_A \oplus T := D_A \oplus d^* + Id_{H^0} \oplus \pi_{\text{harm}} \oplus d^+ : \Gamma(\mathbb{S}^+) \oplus \Omega^1(X) \oplus H^0(X;\mathbb{R}) &\to \Gamma(\mathbb{S}^-) \oplus \Omega^0(X) \oplus H^1(X;\mathbb{R}) \oplus \Omega^+(X) \\ (\varphi, a, [f]) &\mapsto (D_A\varphi, d^*(a) + [f], a_{\text{harm}}, d^+(a)) \end{aligned}$$

The Dirac operator associated to the Spin^c-structure $\mathfrak{s}$ defines a virtual complex index bundle $\operatorname{ind}(D)$ over the Picard torus. Observe that the other maps are defined and lands in spaces orthogonal to $\Gamma(\mathbb{S}^\pm)$, therefore we can study their index separately from $\operatorname{ind}(D)$. We start by observing that T is injective. Let $(a, [f])$ be in $\ker T$. We can assume f to be harmonic, and since $\mathcal{H}^0 \perp \operatorname{Im} d^*$, we have that $(a, f) \in \ker T \Rightarrow f = 0$. Now let $(a, 0) \in \ker T$, in particular, using Hodge decomposition we can write, uniquely, $a = a_{\text{harm}} + d^*\omega + dg$. Notice that π_{harm}, d^* and d^+ act independently and exclusively on a_{harm}, dg and $d^*\omega$ respectively, due to the orthogonal decomposition and the fact that they vanish on the other components. From $d^*dg = 0$ we have

$$\begin{aligned} 0 = d^*dg &= \langle d^*dg, g\rangle_{L^2} \\ &= \langle dg, dg\rangle_{L^2} \\ &= \|dg\|_{L^2} \end{aligned}$$

Hence $dg = 0$. Similarly, $0 = d^+d^*\omega$ implies $dd^*\omega = 0$ (Theorem 2.1, L9 in [3]) and by the same reasoning as before $d^*\omega = 0$, hence $(a, [f]) \in \ker T \Leftrightarrow a = 0, f = 0$.
We now look at the cokernel. By the discussion about the Hodge decomposition and how the component of T are independent of each other we can study those components separately. Immediately we see that π_{harm} is surjective so it won't contribute to the cokernel. Again by the Hodge decomposition, $d^* + \mathrm{Id}_{H^0}$ is surjective, while the cokernel of d^+ is by definition

$$\frac{\Omega^+(X)}{\operatorname{Im} d^+} = \mathcal{H}^+ \cong H^+(X;\mathbb{R})$$

proving the claim. □

Let's define the homomorphism of the stable cohomotopy group to $\mathbb{Z}$. An element of the stable cohomotopy group is represented by an equivariant map $f : TE \to S^U$ from the Thom space of a bundle E over $Pic^{\mathfrak{s}}(X)$, where $E - \underline{U} = \operatorname{ind} l$ is the index of the linearisation l of μ. Let Ci denote the mapping cone of the inclusion $i : TE^{S^1} \to TE$ of the S^1-fixed point set. In the long exact sequence associated to the cofiber sequence $TE^{S^1} \to TE \to Ci$,

$$\pi^{-1}_{S^1,\mathcal{U}}(\Sigma^{-U}(TE^{S^1})) \to \pi^0_{S^1,\mathcal{U}}(\Sigma^{-U}Ci) \to \pi^0_{S^1,\mathcal{U}}(\Sigma^{-U}TE) \to \pi^0_{S^1,\mathcal{U}}(\Sigma^{-U}(TE^{S^1})) \tag{3}$$

Claim 2.7. $\pi^{-1}_{S^1,\mathcal{U}}(\Sigma^{-U}(TE^{S^1})) = 0$, *similarly* $\pi^0_{S^1,\mathcal{U}}(\Sigma^{-U}(TE^{S^1})) = 0$.

Proof. We observe that by definition $\pi^{-1}_{S^1,\mathcal{U}}(\Sigma^{-U}(TE^{S^1})) = [S^1 \wedge TE^{S^1}, S^U]$ where the U comes from the representation of the index of l as formal difference $E - \underline{U}$. Now recall that the action of S^1 on $\Gamma(\mathbb{S})$ is free, given by multiplication, hence they don't contribute to the fixed point sets. Therefore TE^{S^1} is simply the component of the Thom space of the index arising from the kernel of T since the action there is trivial by definition. Since we proved that the map is injective, $TE^{S^1} = Pic^{\mathfrak{s}}(X)^{S^1}_+$ which has dimension b_1. Similarly we see that the fixed points in S^U have dimension b^+ since only the coker of T contributes to the fixed points. Now using the assumption $b^+ - b_1 - 1 > 0$ by cellular approximation we have the claim for the first group. The proof for the last group is done in the same way. □

So the map $\mu \in \pi^0_{S^1,\mathcal{U}}(\Sigma^{-U}TE)$ can be described by a cohomotopy element of $\Sigma^{-U}Ci$. The Hurewicz image $h(\mu)$ of this element in equivariant Borel-cohomology lies in the relative group $H^0_{S^1}(\Sigma^{-U}TE, \Sigma^{-U}TE^{S^1})$. The S^1-action on the pair of spaces (TE, TE^{S^1}) is relatively free. So its equivariant cohomology group identifies with the singular cohomology $H^*(TE/S^1, TE^{S^1})$ of the quotient. After replacing TE^{S^1} by a tubular neighbourhood, this is the singular cohomology of a connected manifold relative to its boundary.
In fact, using the injectiviness of T and the fact that the action is free on the positive spinors, we see that after we remove the zero section the action on each fibre is free and given by multiplication. Notice that we can write $E \setminus 0 \simeq S(E) \times \mathbb{R}_{>0}$, which can be rescaled in $E \setminus 0 \simeq S(E) \times (0,1)$. Since the action of S^1 doesn't change the norm of our vectors, the action of S^1 is free on the first factor and trivial on the second. When we consider the orbit we get $(E \setminus 0)_{S^1} \simeq P(E) \times \mathbb{R}_{>0}$. After we include back the fixed point set it's immediate to see that we get $(TE/S^1, TE^{S^1}) \sim (P(E) \times I, P(E) \times \{0,1\})$ proving the claim.

Claim 2.8. *There is a generator in the top degree of* $H^*_{S^1}(\Sigma^{-U}TE, \Sigma^{-U}TE^{S^1})$

Proof. An orientation of $Pic^{\mathfrak{s}}(X)$ together with the standard orientation of complex vector bundles defines an orientation $[TE/S^1, TE^{S^1}] =: [TE]_{S^1}$ in the top cohomology of this *manifold*. Similarly, the chosen homology orientation of X and the orientation of $Pic^{\mathfrak{s}}(X)$ determine the orientation of U and thus a generator $\Sigma^{-U}[TE]_{S^1}$ of the graded cohomology group $H^*_{S^1}(\Sigma^{-U}TE, \Sigma^{-U}TE^{S^1})$ in its top degree $* = k$. □

Claim 2.9. *The top degree k is such that $k = ind_{\mathbb{R}}(D_A) + (b_1 - b^+ - 1)$ A is any chosen $Spin^c$-connection.*

Proof. We have to compute the dimension of the manifold whose cohomology ring is $H^*(\Sigma^{-U}TE/S^1, \Sigma^{-U}TE^{S^1})$. For simplicity assume that the Dirac operator is surjective. This means that $\dim_{\mathbb{C}} E = \text{ind}_{\mathbb{C}} D$. Therefore we have $\dim_{\mathbb{R}} P(E) = 2(\text{ind}_{\mathbb{C}} D - 1) + b_1 = \text{ind}_{\mathbb{R}} D - 2 + b_1$ since we have to factor in the dimension of the base space (the Picard torus). Now $\dim_{\mathbb{R}}(P(E) \times I) = \text{ind}_{\mathbb{R}} D - 1 + b_1$. Now we have to desuspend which lowers the degree in cohomology by $\dim \mathcal{H}^+ = b^+$ (recall that D is assumed to be surjective), hence we get $k = \text{ind}_{\mathbb{R}}(D) + (b_1 - b^+ - 1)$. If D is non surjective the result still holds since the difference between the coker and ker of D is constant and equal to $\text{ind}_{\mathbb{C}} D$, and first we add the dimension of the kernel and then when we desuspend we subtract the dimension of the cokernel, hence only the index of D really matter. □

This is great news, since the classical Seiberg-Witten invariants are obtaining as the pairing in this exactly same degree.
Observe that $H^*_{S^1}(\Sigma^{-U}TE, \Sigma^{-U}TE^{S^1})$ is a module over the coefficient ring $H^*_{S^1}(*) \cong \mathbb{Z}[t]$, where t has degree 2. Now let us define

$$\mathfrak{sw}'_{X,\sigma} = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \langle t^{k/2} h(\mu), \Sigma^{-U}[TE]_{S^1} \rangle & \text{if } k \text{ is even} \end{cases}$$

Claim 2.10. $\mathfrak{sw}'_{X,\sigma} = \mathfrak{sw}_{X,\sigma}$

Proof. We only need to prove that when k is even, $\langle t^{k/2} h(\mu), \Sigma^{-U}[TE]_{S^1} \rangle = \langle t^{k/2}, [M_\mu] \rangle$ because this is the original definition of the Seiberg-Witten invariants. But this is a consequence of the geometric interpretation of the Hurewicz map and Poincarè duality. In fact $\Sigma^U h(\mu) = \mu^*[S^U] = \mu^*[U, U \setminus 0] = [\nu_{M_\mu}, \nu_{M_\mu} \setminus M_\mu]$ (naturality of the Thom class). But now

$$[\nu_{M_\mu}, \nu_{M_\mu} \setminus M_\mu] \frown [TE]_{S^1} = [M_\mu]$$

(Ex. 11-C in Milnor's Characteristic classes, recall that $M_\mu = \mu^{-1}(0)$ - with a little abuse of notation) □

References

[1] Stefan Bauer, *Refined Seiberg-Witten invariants*, Donaldson S., Eliashberg Y., Gromov M. (eds) Different Faces of Geometry. International Mathematical Series, vol 3. Springer, Boston, MA

[2] Stefan Bauer, Mikio Furuta *A stable cohomotopy refinement of Seiberg-Witten invariants: I* Invent. math. (2004) 155: 1.

[3] Timothy Perutz, *A course on Seiberg-Witten theory and* 4*-manifold topology* Lecture notes

[4] Borisovich, Zvyagin, Sapronov, *Non-Linear Fredholm maps and the Leray-Schauder Theory*, Russian Math. Surveys, 32:4 (1977), 1-54

[5] Lawson, Michelsohn *Spin Geometry* Princeton University Press, 1989

[6] J. W. Morgan *The Seiberg-Witten Equations and Applications to the Topology of Smooth Four-Manifolds*