A proof of the non-squeezing theorem

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Abstract

We will give proof of the non-squeezing theorem by using pseudo-holomorphic curves and Gromov-Witten flavoured techniques. We will blackbox some analytical facts about the smooth structure of some moduli spaces of solutions since they are lengthy but standard argument that can be found on the standard literature about this topic. Modulo these results, no particular prior knowledge is assumed. I will follow closely an exposition of John Pardon (errors are all mine), pictures were made by Jackson van Dyke and used here with his permission.

1 Some background

Definition 1.1. A symplectic form on a manifold M^{2n} is a closed 2-form ω which, when viewed as a pairing

$$\omega : TM \otimes TM \to \mathbb{R}$$

is non-degenerate (i.e. the induced map $TM \to TM^{\vee}$ is an isomorphism.) and skew-symmetric.

Example 1.2. $\mathbb{R}^{2n}$ with standard basis $\{x_1, \cdots, x_n, y_1, \cdots, y_n\}$ is a symplectic manifold with the symplectic form

$$\omega_{\text{std}} := \sum_{i=1}^{n} dx_i \wedge dy_i \tag{1}$$

Theorem 1.3 (Darboux). *For any symplectic manifold M and point $p \in M$, there exists a function $f : U \to M$, (diffeomorphic onto its image), where $U \subseteq \mathbb{R}^{2n}$ is open, $p \in f(u)$, and $f^*(\omega_M) = \omega_{std}$.*

Define the symplectomorphisms $\text{Diff}_0(M, \omega)$ to consist of the diffeomorphisms preserving the symplectic form. Of course this is contained inside $\text{Diff}_0(M, \omega^n)$, which are volume preserving. Explicitly, the symplectomorphisms consist of the identity component of

$$\{f : M \to M \mid f^*\omega = \omega\}_0 \subseteq \{f : M \to M \mid f^*(\omega^n) = \omega^n\} \tag{2}$$

2 The non squeezing theorem

We introduce the following notation:

$$B^{2n}(r) := \left\{(x_1, \cdots, x_n, y_1, \cdots, y_n) \in \mathbb{R}^{2n} \mid \sum x_i^2 + y_i^2 \leq r^2\right\}$$
$$B^2(R) \times \mathbb{R}^{2n-2} := \left\{(x_1, \cdots, x_n, y_1, \cdots, y_n) \mid x_1^2 + y_1^2 \leq R^2\right\}$$

The theorem we will try to prove is the following

Theorem 2.1 (Eliashberg, Caley-Zehnder, Gromov - all independently). *If there exists a symplectic embedding:*

$$B^{2n}(r) \to B^2(R) \times \mathbb{R}^{2n-2}$$

then $r \leq R$

2.1 Pseudo-holomorphic curves

To prove Theorem 2.1 we introduce the following notions:

Definition 2.2. An almost complex structure (acs) on M^{2n} is an endomorphism $J\colon TM \to TM$ such that $J^2 = -\operatorname{Id}$.

Equivalently this is putting the structure of a $\mathbb{C}$-vector space on TM, and then J is just multiplication by i.
An acs J is said to be tamed by ω iff $\omega(v, Jv) > 0$, for all nonzero $v \in TM$. J is said to be compatible with ω iff $(v, w) \mapsto \omega(v, Jw)$ is a Riemannian metric, that is, it is symmetric and positive definite. Clearly compatible implies tame. We will work with this assumption all time.

Definition 2.3. A pseudo-holomorphic (or J-holomorphic, or (J, j_Σ)-holomorphic) curve is a smooth map $u\colon \Sigma \to M$, where Σ is a Riemann surface with the canonical complex structure j_Σ such that

$$du \circ j_\Sigma = J \circ du$$

2.2 Proof of the non-squeezing theorem

Assume that there exists a symplectic embedding

$$f\colon B^{2n}(r) \to B^2(R) \times \mathbb{R}^{2n-2}$$

WLOG we can assume that $\operatorname{Im}(f)$ is contained in a compact subset of $\operatorname{Int}(B^2(R) \times \mathbb{R}^{2n-2})$. In fact, assume that the image of $B^{2n}(R)$ is not contained in the interior (hence it touches $\partial B^2(R)$), then by restricting to $B^{2n}(r') \subset B^{2n}(r)$ (i.e. $r' < r$) then we can assume that the image lies in the interior (it's an embedding!). Notice that if we prove that for any $r' < r$, $r' < R$ then it must be that $r < R$. This is going to be our strategy
Notice that we immediately have that $\operatorname{Im}(f)$ is a compact subset of the interior of the cylinder, hence there must be an $N > 0$ such that the we have an embedding

$$f\colon B^{2n}(r) \hookrightarrow B^2(R) \times \left(\mathbb{R}^{2n-2}/N\mathbb{Z}^{2n-2}\right) \tag{3}$$

We now write $\mathbb{R}^{2n-2}/N\mathbb{Z}^{2n-2} = T^{2n-2}$, and further embed

$$f\colon B^{2n}(r) \hookrightarrow B^2(R) \times \left(\mathbb{R}^{2n-2}/N\mathbb{Z}^{2n-2}\right) \to S^2 \times T^{2n-2} \tag{4}$$

where S^2 is the sphere with the same area of the disk $B^2(r)$ obtained by collapsing the boundary. The fact that it is still an embedding is due to the fact we assumed that $\operatorname{Im}(f)$ is contained in the interior of the cylinder. All together we have a symplectic embedding

$$f\colon B^{2n}(r) \hookrightarrow \left(S^2 \times T^{2n-2}, \omega_{S^2} + \omega_{T^{2n-2}}\right) \tag{5}$$

Now equip $S^2 \times T^{2n-2}$ with a product acs $J = J_{S^2} \oplus J_{T^{2n-2}}$, so it is biholomorphic to $\mathbb{C}P^1 \times \left(\mathbb{C}^{n-1}/\mathbb{Z}^{2n-2}\right)$ as a complex manifold. Now consider the following moduli space

$$\mathcal{M}_{0,1}(S^2 \times T^{2n-2}, 0, J) := \{u\colon \mathbb{C}P^1 \to S^2 \times T^{2n-2} \mid du \text{ is } \mathbb{C} \text{ linear}\}/\sim \tag{6}$$

where $u \sim u \circ \phi \mid \phi\colon \mathbb{C}P^1 \to \mathbb{C}P^1$ is an element in $\mathrm{PGL}_2(\mathbb{C})$ (i.e. an automorphism of the Riemann sphere) which preserve the origin $0 \in \mathbb{C} \subset \mathbb{C} \cup \{\infty\}$.
If we have a map

$$u\colon \mathbb{C}P^1 \to \mathbb{C}P^1 \times \left(\mathbb{C}^{n-1}/\mathbb{Z}^{2n-2}\right) \tag{7}$$

this splits as:

$$u = (u_1, u_2) \quad u_1\colon \mathbb{C}P^1 \to \mathbb{C}P^1 \quad u_2\colon \mathbb{C}P^1 \to \mathbb{C}^{n-1}/\mathbb{Z}^{2n-2}$$

Now, since $\pi_1(\mathbb{C}P^1)=0$, we always have a lift

$$\begin{tikzcd} & \mathbb{C}^{n-1} \arrow[d, "\pi"] \\ \mathbb{C}P^1 \arrow[r, "u_2"] \arrow[ur, dashed] & \mathbb{C}^{n-1}/\mathbb{Z}^{2n-2} \end{tikzcd} \tag{8}$$

Therefore by Liouville's theorem, u_2 is constant.
We have an obvious foliation by holomorphic $\mathbb{C}P^1$ as in fig. 1. with respect to this acs

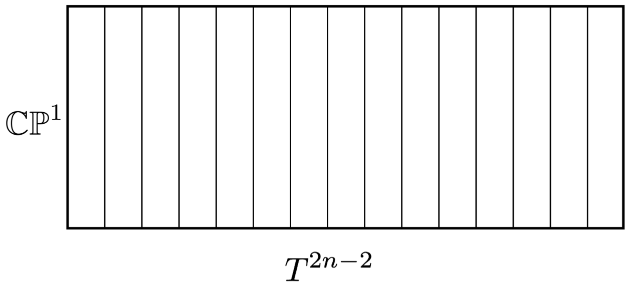

Figure 1: Foliation by holomorphic $\mathbb{C}P^1$.

$$J_0 = J_{\mathbb{C}P^1} \oplus J_{\mathbb{C}^{n-1}/\mathbb{Z}^{2n-2}} \tag{9}$$

Now let

$$\mathcal{M}_{0,1}(S^2\times T^{2n-2},0,d[S^2]\times[\text{pt.}]) \subseteq \mathcal{M}_{0,1}(S^2\times T^{2n-2},0) \tag{10}$$

be the subset of pseudo-holomorphic curves with $u_*[\mathbb{C}P^1]=d[S^2]\times[\text{pt.}]$. In particular, by taking $d=1$ we focus on $\mathcal{M}_{0,1}(S^2\times T^{2n-2},0,[S^2]\times[\text{pt.}])$. We have an evaluation map

$$\begin{aligned} \text{ev}\colon \mathcal{M}_{0,1}(S^2\times T^{2n-2},0,[S^2]\times[\text{pt.}]) &\to S^2\times T^{2n-2} \\ [u] &\mapsto u(0) \end{aligned}$$

Let us call J_1 the acs on $S^2\times T^{2n-2}$ which agrees with f_*J_{std} over $\text{Im}(f)$.

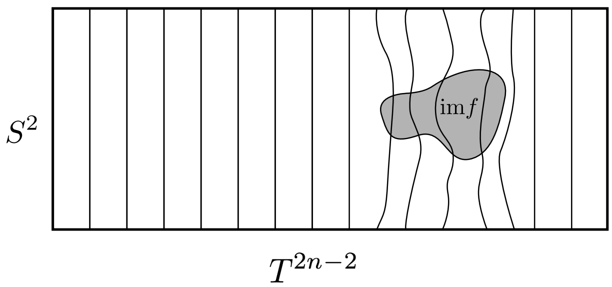

Figure 2: Foliation by J_1-holomorphic $\mathbb{C}P^1$.

It's a classical result that the space of ω-compatible almost complex structure is nonempty and contractible. Therefore we can choose a path of complex structures $\{J_t\}_{t\in[0,1]}$ between J_0 and J_1. Now the collection of $t\in[0,1]$, such that 1-pointed genus 0, curve in $S^2\times T^{2n-2}$ with respect to J_t representing the class $[S^2]\times[\text{pt.}]$ projects as

$$\begin{array}{c}\mathcal{M}_{0,1}(S^2 \times T^{2n-2}, 0, [S^2] \times [\text{pt.}], \{J_t\}) \\ \Big\downarrow \pi \\ [0,1]\end{array}$$

To continue we need to appeal to some big non-trivial results about moduli spaces of holomorphic curves:

1. Gromov compactness: π is proper
2. For generic choice of J_1 and $\{J_t\}_t$, the total space space is a manifold with boundary $\pi^{-1}(0) \cup \pi^{-1}(1)$.

Claim 2.4. *We have that the evaluation map induces a diffeomorphism* $\pi^{-1}(0) \cong S^2 \times T^{2n-2}$

Proof. We will focus on proving that the evaluation map is a bijection and we will deal with the topology later. First notice that fig. 1 basically proves that the evaluation map is surjective: for every $(x, y) \in S^2 \times T^{2n-2}$, the embedding at *height* y

$$\begin{aligned}\mathbb{C}P^1 &\to S^2 \times T^{2n-2} \\ x &\mapsto (x, y)\end{aligned}$$

provides a perfectly nice J_0-hol sphere with marked point $(0, y)$. By using elements in $PGL_2(\mathbb{C})$ we can manage to send 0 to every point (x, y). This proves that the evaluation map is surjective. Injectivity requires a little bit more of work: first notice that we are working with degree 1 maps $u\colon \mathbb{C}P^1 \to \mathbb{C}P^1$ thanks to the observation in 8. Being degree 1, u must be surjective. At every regular point u must be injective as well, since they are orientation preserving and the sum (with sign) of the preimages of any regular value must be 1. Since we are working with the standard (i.e. integrable) a.c.s. u is holomorphic, therefore is an open map. This implies that u is injective everywhere, hence a biholomorphism. Therefore if $u, u'\colon \mathbb{C}P^1 \to \mathbb{C}P^1$ are two J_0-holomorphic curves with $u(0) = u'(0) = *$, then we have

$$\begin{aligned}u &= u' \circ u'^{-1} \circ u \\ &= u' \circ \phi\end{aligned}$$

for $\phi \in PGL_2(\mathbb{C})$ with $\phi(0) = 0$, proving that they belong to the same equivalence class. By choosing the correct topology on our moduli space the evaluation map can be shown to be continuous, hence an homeomorphism (recall we assumed our moduli space to be compact). Smoothness is done similarly. □

Now consider the evaluation map ev. We know $\text{ev}_*[\pi^{-1}(0)] = [S^2 \times T^{2n-2}]$, but we have this cobordism, so actually $\text{ev}_*[\pi^{-1}(1)] = \text{ev}_*[\pi^{-1}(0)]$. In particular this implies that

$$\text{ev}\colon \mathcal{M}_{0,1}(S^2 \times T^{2n-2}, 0, [S^2] \times [\text{pt.}], J_1) \to S^2 \times T^{2n-2}$$

is surjective.

Consider a holomorphic curve $u\colon \mathbb{C}P^1 \to S^2 \times T^{2n-2}$ which passes through $f(0)$ where f is our embedding. We have therefore actually produced a curve $v\colon \Sigma \to B^{2n}(r)$ (by pull back) passing through the origin, where Σ is as in the following diagram:

$$\begin{tikzcd}\Sigma := u^{-1}\left(f(B^{2n}(r))\right) \arrow[r, hook] \arrow[d, "v"] & \mathbb{C}P^1 \arrow[d, "u"] \\ B^{2n}(r) \arrow[r, hook, "f"] & S^2 \times T^{2n-2}\end{tikzcd}$$

where v is proper, since it's the pullback of the proper map u. The picture here is as in fig. 3

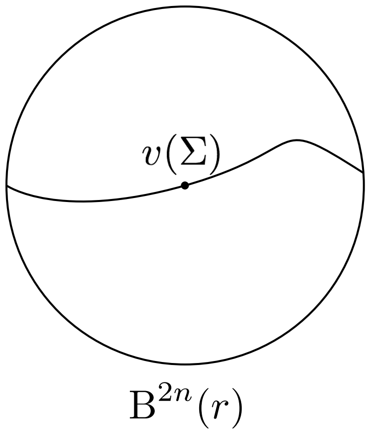

Figure 3: The curve $v(\Sigma)$ on $B^{2n}(r)$

Now we have the following inequalities:

$$\begin{aligned} \text{Area}(v(\Sigma)) &= \int_{\Sigma} v^*\omega && (11)\\ &\leq \int_{\mathbb{C}P^1} u^*\omega && (12)\\ &= \langle u_*[\mathbb{C}P^1], [\omega]\rangle && (13)\\ &= \text{Area}(S^2) = \text{Area}\big(B^2(R)\big) = \pi R^2 && (14)\end{aligned}$$

Now the last ingredient is the following:

Proposition 2.5 (Monotonicity)**.** *If $v\colon \Sigma \to B^{2n}(r) \subseteq \mathbb{C}^n$ is a proper, non-constant, holomorphic (with respect to J_{std}) map covering the origin, then*

$$\int_{\Sigma} v^*\omega \geq \pi r^2 \tag{15}$$

We will divide the proof in three steps

Claim 2.6. *For any orientable surface $\Gamma(r) \subset B^{2n}(r)$ whose boundary lies in $S^{2n-1}(r)$ we have*

$$\frac{d}{dr}A(r) \geq L(r) \geq \frac{2A(r)}{r} \tag{16}$$

where $A(r)$ is the symplectic area of $\Gamma(r)$ and $L(r)$ the length of $\partial\Gamma(r) = \gamma(r)$.

Proof. Since the disk is an exact symplectic manifold, with Liouville form $\lambda = \frac{1}{2}x_i dy_i - y_i dx_i$ we have, by Stokes, that

$$A(r) := \int_{\Gamma(r)} \omega = \int_{\gamma(r)} \lambda \tag{17}$$

Therefore we can divide by r and obtain

$$\frac{A(r)}{r} = \int_{\gamma(r)} \frac{1}{r}\lambda \tag{18}$$

$$= \int_{\gamma(r)} \omega(\tfrac{1}{r}V_\lambda, -) \tag{19}$$

$$= \int_{[0,2\pi]} \omega(\tfrac{1}{r}V_\lambda, \dot{\gamma(r)})dt \tag{20}$$

$$= \int_{[0,2\pi]} \langle -J\tfrac{1}{r}V_\lambda, \dot{\gamma(r)}\rangle dt \tag{21}$$

$$\leq \frac{1}{2}L(r) \tag{22}$$

where in Passage (19) we used the definition of the Liouville vector field. In (21) we used the fact that the metric is chosen to be compatible with the symplectic structure. The last passage (22) we made use of the fact that the Liouville vector field is radially expanding from the origin of the disk and we have a normalization factor of $1/r$ that we are carrying around. The 2 comes from the fact that the Liouville form has a $1/2$ in its definition. By rotating such vector field by $\pi/2$ (effect of $-J$) we get the upper bound.
The other inequality is proved as followed. Let $\epsilon > 0$, we want to estimate $A(r+\epsilon) - A(r)$ from below. Using the tubular neighbourhood theorem we can find a local parametrization for $\Gamma(r)$ around $\gamma(r)$ (which cover $\gamma(r+\epsilon)$ as well). Let us denote it as $\Gamma(\theta, t)$ where $\theta \in [0, 2\pi]$ is a parametrization for $\gamma(r)$ (and its translates) and $t \in [0, 1]$ is the normal direction (remember that the tubular neighbourhood of $\gamma(r) \subset \Gamma(r)$ is trivial since it's orientable.

Notice that we can compute the area of the *shell* $A(r+\epsilon) - A(r)$ as the integral

$$A(r+\epsilon) - A(r) = \int_{[0,2\pi]\times[0,1]} \Gamma(\theta, t)d\theta dt \tag{23}$$

Now via Fubini's theorem we have

$$A(r+\epsilon) - A(r) = \int_{[0,2\pi]} \int_{[0,1]} \Gamma(\theta, t)d\theta dt \tag{24}$$

$$\leq L(r)\epsilon \tag{25}$$

since every vertical segment (i.e. $\Gamma(\theta, [0, 1]) \geq \epsilon$ since that's the shortest path from a point on $S(r)$ to a point in $S(r+\epsilon)$. Therefore by taking the limit for $\epsilon \to 0$ we have the required inequality □

Claim 2.7. *If a loop in $S^{2n-1}(r)$ has length strictly less than $2\pi r$, then it's contained in some open hemisphere of the sphere*

Proof. See `https://math.stackexchange.com/questions/1805648/` □

Proof of Prop. 2.5. It's a standard fact about J-holomorphic curves that they are the area minimizing surfaces among surfaces that shares the same boundary. Moreover their symplectic area coincides with the geometrical one. Since by construction we know that v passes through the origin, by Claim 2.7 we know it must have $L(r) \geq 2\pi r$. by integrating over $[0, r]$ the left inequality given in Claim 2.6 we get that $A(r) \geq \pi r^2$, concluding the proof □

Now the upper bound given by (14) and the lower bound given by Gromov's monotonicity (Prop 2.5) implies that $r \leq R$, concluding the proof of the non-squeezing theorem.