

Complex oriented cohomology, generalized orientation and Thom isomorphism

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The material written here is heavily influenced by Kochman's account [1] on complex oriented cohomology theory. Any mistakes in the notes are my own: if you find anything unclear or wrong write me an e-mail at `pedrotti.riccardo@math.utexas.edu`.

1 Spectra, homotopy group of spectra and ring spectra

Let us denote $SX = S^1 \wedge X$ for a pointed space X .

Definition 1.0.1. A *spectrum* E is a sequence $\{E_n, s_n\}$, $n \in \mathbb{Z}$, of CW-complexes E_n and CW-embeddings $s_n: SE_n \rightarrow E_{n+1}$.

Let us give a pair of examples: Let X be your favourite (pointed) CW-complex. Let us define the spectrum $\Sigma^\infty X$ as follows

$$(\Sigma^\infty X)_n := \begin{cases} * & \text{if } n < 0 \\ S^n X & \text{if } n \geq 0 \end{cases} \quad (1)$$

and the maps $s_n: S(S^n X) \rightarrow S^{n+1} X$ is the identity map.

A very important instance of this construction is given by the case $X = S^0$. The resulting spectrum, $S^\infty S^0$ is often denoted by \mathbb{S} and called the sphere spectrum.

Another useful construction is the following: given a spectrum $E = \{E_n, s_n\}$ and an integer $k \in \mathbb{Z}$ we can define the *shifted* spectrum $\Sigma^k E$ as follows:

$$(\Sigma^k E)_n = E_{n+k} \quad (2)$$

and as maps $s'_n: S(\Sigma^k E)_n \rightarrow (\Sigma^k E)_{n+1}$ we take s_{k+n} .

We want to find a suitable definition for the category of spectra. For technical reason, it's convenient to give the following definition:

Definition 1.0.2. A *cell* of a spectrum E is a sequence $\{e, Se, \dots, S^k e, \dots\}$ where e is a cell of any E_n such that e is not the suspension of any cell of E_{n-1} . If e is a cell of E_n of dimension d then the dimension of the cell $\{e, Se, \dots, S^k e, \dots\}$ is $d - n$. A subspectrum F of a spectrum E is *cofinal* (in E) if every cell of E is eventually in F , i.e. for every cell $e \in E_n$, there exists m such that $S^m e$ belongs to F_{n+m} .

Definition 1.0.3. • Let $E = \{E_n, s_n\}$ and $F = \{F_n, t_n\}$ be two spectra. A *map* f from E to F is a family of pointed cellular maps $f_n: E_n \rightarrow F_n$ such that the following diagram commutes:

$$\begin{array}{ccc} SE_n & \xrightarrow{s_n} & E_{n+1} \\ \downarrow Sf_n & & \downarrow f_{n+1} \\ SF_n & \xrightarrow{t_n} & F_{n+1} \end{array}$$

- Let E, F two spectra as above. Consider the set \mathcal{A} of pairs (f', E') where E' is cofinal in E and $f': E' \rightarrow F$ is a map. Consider the equivalence relation \sim on \mathcal{A} such that $(f', E') \sim (f'', E'')$ if and only if $f'|_B = f''|_B$ for some $B \subset E' \cap E''$ with B cofinal in E . Every such equivalence class is called a *morphism from E to F* .

We can therefore form the category \mathcal{S} whose objects are spectra as defined above and morphisms of spectra as arrows.

Definition 1.0.4. • Two maps $g_0, g_1: E \rightarrow F$ of spectra are called *homotopic* if there exists a map $G: E \wedge I^+ \rightarrow F$ (called a *homotopy*) such that G coincides with g_i on the subspectrum $E \wedge \{i, *\}$, $i = 0, 1$ of E . In this case we write $g_0 \simeq g_1$.

- Two morphisms $\varphi_0, \varphi_1: E \rightarrow F$ of spectra are called *homotopic*, if there exists a cofinal subspectrum E' of E and two maps $g_i: E' \rightarrow F$, $g_i \in \varphi_i$ for $i = 0, 1$, such that $g_0|_{E'} \simeq g_1|_{E'}$.

We can define the category \mathcal{HS} , with spectra as objects and sets $[E, F]$ as sets of morphisms. Isomorphisms of \mathcal{HS} are called *equivalences*, and we use the notation $E \simeq F$ when E is equivalent to F .

Proposition 1.0.5. *The spectra $S^1 \wedge E$ and ΣE are equivalent*

Proof. See [4] 8.26 □

Definition 1.0.6. Let \mathbb{S} be the sphere spectrum, we define the k -th homotopy group of a spectrum E as follows:

$$\pi_k(E) = [\Sigma^k \mathbb{S}, E]$$

It's easy to see that $\pi_k(E) = \text{colim}_{N \rightarrow \infty} \pi_{k+N}(E_N)$ where the directed limit is that of the directed system

$$\cdots \longrightarrow \pi_{k+N}(E_N) \xrightarrow{S} \pi_{k+N+1}(SE_N) \xrightarrow{s_{N+1}} \pi_{k+N+1}(E_{N+1}) \longrightarrow \cdots$$

In particular if $E = \Sigma^\infty X$ then the homotopy group of E is the stable homotopy group of the space X .

Definition 1.0.7. A *prespectrum* is a family $\{X_n, t_n\}$, $n \in \mathbb{Z}$ of pointed spaces X_n and pointed maps $t_n: SX_n \rightarrow X_{n+1}$.

Lemma 1.0.8. *For every prespectrum $\{X_n, t_n\}_n$, there exist a spectrum $E = \{E_n, s_n\}$ and pointed homotopy equivalences $f_n: E_n \rightarrow X_n$ such that the diagram*

$$\begin{array}{ccc} SE_n & \xrightarrow{Sf_n} & SX_n \\ \downarrow s_n & & \downarrow t_n \\ E_{n+1} & \xrightarrow{f_{n+1}} & X_{n+1} \end{array}$$

commutes.

Proof. See [4] Proposition 8.3 □

Theorem 1.0.9.¹ *There is a construction which assigns to spectra E, F a certain spectrum denoted by $E \wedge F$. This construction is called the smash product $E \wedge F$ of spectra and has the following properties:*

1. *It is a covariant functor of each of its arguments.*
2. *There are natural equivalences:*

$$\begin{aligned} a &= a(E, F, G): (E \wedge F) \wedge G \rightarrow E \wedge (F \wedge G) \\ \tau &= \tau(E, F): E \wedge F \rightarrow F \wedge E \\ l &= l(E): \mathbb{S} \wedge E \rightarrow E \\ r &= r(E): E \wedge \mathbb{S} \rightarrow E \\ S &= S(E, F): \Sigma E \wedge F \rightarrow \Sigma(E \wedge F) \end{aligned}$$

3. *For every spectrum E and CW-complex X , there is a natural equivalence $e = e(E, X): E \wedge X \rightarrow E \wedge \Sigma^\infty X$. In particular $\Sigma^\infty(X \wedge Y) \simeq \Sigma^\infty X \wedge \Sigma^\infty Y$ for every pair of CW-complexes X, Y*
4. *If $f: E \rightarrow F$ is an equivalence then $f \wedge \text{Id}_G: E \wedge G \rightarrow F \wedge G$ is.*
5. *Let $\{E_\lambda\}$ be a family of spectra, and let $i_\lambda: E_\lambda \rightarrow \bigvee_\lambda E_\lambda$ be the inclusions. Then the morphism*

$$\{i_\lambda \wedge \text{Id}\}: \bigvee_\lambda (E_\lambda \wedge F) \rightarrow (\bigvee_\lambda (E_\lambda)) \wedge F$$

is an equivalence

¹Theorem 2.1 page 45 on [6]

6. if $A \xrightarrow{f} B \xrightarrow{g} C$ is a cofiber sequence of spectra, then so is the sequence

$$A \wedge E \xrightarrow{f \wedge \text{Id}} B \wedge E \xrightarrow{g \wedge \text{Id}} C \wedge E$$

for every spectrum E .

We are ready now to give the following definition:

Definition 1.0.10 (Ring Spectrum). ² A ring spectrum is a triple (E, μ, ι) where E is a spectrum, $\mu: E \wedge E \rightarrow E$ and $\iota: \mathbb{S} \rightarrow E$ are morphisms such that the following diagrams commute up to homotopy:

- Associativity:

$$\begin{array}{ccc} (E \wedge E) \wedge E & \xrightarrow{\mu \wedge 1} & E \wedge E \\ \downarrow a & & \searrow \\ E \wedge (E \wedge E) & \xrightarrow{1 \wedge \mu} & E \wedge E \xrightarrow{\mu} E \end{array}$$

where a is a natural equivalence given by definition of the smash product of spectra.

- Unitary:

$$\begin{array}{ccccc} \mathbb{S} \wedge E & \xrightarrow{\iota \wedge 1} & E \wedge E & \xleftarrow{1 \wedge \iota} & E \wedge \mathbb{S} \\ \downarrow l & & \downarrow \mu & & \downarrow r \\ E & \xrightarrow{\text{Id}} & E & \xleftarrow{\text{Id}} & E \end{array}$$

Where l, r are natural equivalences given by definition of the smash product of spectra.

The ring spectrum is commutative if the following diagram commutes up to homotopy:

$$\begin{array}{ccc} E \wedge E & \xrightarrow{\tau} & E \wedge E \\ \downarrow \mu & & \downarrow \mu \\ E & \xrightarrow{\text{Id}} & E \end{array}$$

Where τ twists the factors of the smash product.

A morphism of ring spectra $\varphi: (E, \mu, \iota) \rightarrow (E', \mu', \iota')$ is a morphism $\varphi: E \rightarrow E'$ s.t. the following diagrams commute up to homotopy:

$$\begin{array}{ccc} E \wedge E & \xrightarrow{\varphi \wedge \varphi} & E' \wedge E' \\ \downarrow \mu & & \downarrow \mu' \\ E & \xrightarrow{\varphi} & E' \end{array} \quad \begin{array}{ccc} \mathbb{S} & \xrightarrow{\iota} & E \\ \downarrow \text{Id} & & \downarrow \varphi \\ \mathbb{S} & \xrightarrow{\iota'} & E' \end{array}$$

Examples of ring spectra are provided by spectra X equipped with maps:

$$X_{n_1} \wedge X_{n_2} \rightarrow X_{n_1+n_2}$$

which are suitably associative and unital. This is the case for example of the sphere spectrum (whose multiplication is the classical smash product) and the Thom spectra, which we will see later. Before concluding this section, one has to take this example with a grain of salt. In order to be sure that this smash product really factor through the homotopy category then one needs some additional compatibility conditions.

Proposition 1.0.11. *Up to homotopy there is only one morphism of ring spectra $\mathbb{S} \rightarrow X$, for X any ring spectrum.*

²Definition 2.12 page 51 on [6]

Before giving the proof, notice that working in the category of Ring Spectra is crucial here. In fact if we are allowed to consider map which are not morphisms of ring spectra, then the proposition is false, for example, for any non trivial ring R , $\pi_0(HR) \cong R \not\cong 0$.

Proof. Notice that \mathbb{S} is a ring spectrum, where the product is the degree-wise smash product and the unit is the identity $\text{Id}: \mathbb{S} \rightarrow \mathbb{S}$. Take any morphism of ring $\varphi: \mathbb{S} \rightarrow X$. Then last diagram given by the axiom of ring morphism with $E = \mathbb{S}$, $E' = X$ and $\iota = \text{Id}$ gives the proof:

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{\text{Id}} & \mathbb{S} \\ \downarrow \text{Id} & & \downarrow \varphi \\ \mathbb{S} & \xrightarrow{\iota} & X \end{array}$$

□

Homology and Cohomology induced by a Spectrum

Let E be an arbitrary spectrum.

- Define covariant functors $\tilde{\mathbb{E}}_n: \mathbf{CW}^\bullet \rightarrow \mathbf{Ab}$ where $\tilde{\mathbb{E}}_n(X) := \pi_n(E \wedge \Sigma^\infty X)$ for every $X \in \mathbf{CW}^\bullet$ and $\tilde{\mathbb{E}}_n(f) := \pi_n(\text{Id}_E \wedge \Sigma^\infty f)$ for every morphism $f: X \rightarrow Y$ of pointed CW complexes. One can verify that it is a *reduced* homology theory and that it is additive. See [6] page 63 construction 3.13.
- One can build cohomology theories as well, by defining contravariant functors $\tilde{\mathbb{E}}^n: \mathbf{CW}^\bullet \rightarrow \mathbf{Ab}$ by setting $\tilde{\mathbb{E}}^n(X) := [\Sigma^\infty X, \Sigma^n E]$ for every $X \in \mathcal{S}$ and

$$\tilde{\mathbb{E}}^n(f): [\Sigma^\infty Y, \Sigma^n E] \rightarrow [\Sigma^\infty X, \Sigma^n E] \quad \tilde{\mathbb{E}}^n(f)[g] := [g \circ \Sigma^\infty f]$$

for every $f: X \rightarrow Y$ and $g: \Sigma^\infty Y \rightarrow \Sigma^n E$. As before, one can verify that this defines an additive reduced cohomology theory on \mathbf{CW}^\bullet .

Every morphism $\varphi: E \rightarrow F$ of spectra induces a morphism $\varphi_*: \tilde{\mathbb{E}}_*(-) \rightarrow \tilde{\mathbb{F}}_*(-)$ of homology theories and a morphism $\varphi^*: \tilde{\mathbb{E}}^*(-) \rightarrow \tilde{\mathbb{F}}^*(-)$ of cohomology theories. Here

$$\varphi = \{\varphi_i^X: \tilde{\mathbb{E}}_i(X) \rightarrow \tilde{\mathbb{F}}_i(X)\}, \quad \varphi[f] = [(\varphi \wedge \text{Id}_X) \circ (f)]$$

for every $f: \Sigma^i \mathbb{S} \rightarrow E \wedge X$ for homology and

$$\varphi = \{\varphi_X^i: \tilde{\mathbb{E}}^i(X) \rightarrow \tilde{\mathbb{F}}^i(X)\}, \quad \varphi[f] = [(S^i \varphi) \circ f]$$

for every $f: X \rightarrow \Sigma^i E$ for cohomology. So we have a functor from spectra to (co)homology theories. In particular, equivalent spectra yield isomorphic (co)homology theories.

In order to obtain unreduced (co)homology theory we just define $\mathbb{E}_*(X) := \tilde{\mathbb{E}}_*(X^+)$ and $\mathbb{E}^*(X) := \tilde{\mathbb{E}}^*(X^+)$ as the reduced (co)homology of the space with an adjoint basepoint.

2 Oriented spectrum and the (Co)homology of $\mathbb{C}P^\infty$

The aim of this section is to deal with the notion of an oriented spectrum and its properties. In order to introduce the formal definition let us consider this example.

Example 2.0.1. Let (E, μ, ι) be a ring spectrum. Observe that $\mathbb{C}P^1 \cong S^2$. We want to compute $\tilde{\mathbb{E}}^*(\mathbb{C}P^1)$, and we do it as follows:

$$\tilde{\mathbb{E}}^*(\mathbb{C}P^1) \cong \tilde{\mathbb{E}}^*(S^2) \cong \tilde{\mathbb{E}}^*(\Sigma^2 S^0) \cong \tilde{\mathbb{E}}^{*-2}(S^0) \cong \pi_{*-2} E$$

Since $\tilde{\mathbb{E}}^*$ is a reduced cohomology theory and therefore we can use the suspension isomorphism. If we see the cohomology as a $\pi_* E$ -module, it's free of rank 1 and it's clearly generated, as a module, by the suspension of the unit element $\Sigma^2 \iota$ since the action of the ring $\pi_* E$ on the module is given by the multiplication and since we have a down-shift of two indices.

Definition 2.0.2 (Oriented Spectrum). Let $i: \mathbb{C}P^1 \rightarrow \mathbb{C}P^\infty$ denote the inclusion map. A ring spectrum E with π_*E bounded below is called oriented if there is a class $x^E \in \widetilde{\mathbb{E}}^2(\mathbb{C}P^\infty)$, called orientation class, such that $i^*(x^E) = \Sigma^2\iota$, the canonical generator of $\widetilde{\mathbb{E}}^*(\mathbb{C}P^1)$.

Let us denote with x_E the orientation of a spectrum E . Let $i_n: \mathbb{C}P^n \rightarrow \mathbb{C}P^\infty$ and $i_1^n: \mathbb{C}P^1 \rightarrow \mathbb{C}P^n$ the obvious inclusions. By definition $x_E \in \widetilde{\mathbb{E}}^2(\mathbb{C}P^\infty)$ is such that $i_1^*(x_E) \in \widetilde{\mathbb{E}}^2(\mathbb{C}P^1)$ is a generator of $\widetilde{\mathbb{E}}^*(\mathbb{C}P^1)$ as a π_*E module. To avoid confusion, we will denote the image of these classes to the unreduced cohomology groups with the letter y_E and the same decorations, i.e. $i_n^*(y_E) \in \mathbb{E}^2(\mathbb{C}P^n)$.

Example 2.0.3. We give some examples of oriented spectra:

- We know that for any ring R

$$H^*(\mathbb{C}P^\infty; R) \cong R[x]$$

where $i^*(x)$ is the canonical generator of $H^2(\mathbb{C}P^1; R)$. Therefore the Eilenberg-MacLane spectrum is oriented.

- Let K be the Bott spectrum which defines the complex K -theory:

$$K_n = \begin{cases} BU \times \mathbb{Z} & \text{if } n \text{ even} \\ U & \text{if } n \text{ odd} \end{cases}$$

Let $f: \mathbb{C}P^\infty = BU(1) \rightarrow BU$ (see it as the complex Grassmannian $G_1(\mathbb{C}^\infty)$) be the canonical map and let $\theta: \mathbb{C}P^\infty \rightarrow BU$ be the constant map. Then $f - \theta$ defines a generator x of $\widetilde{K}^2(\mathbb{C}P^\infty)$ which is an orientation of K .

- MU is oriented with orientation $x \in \widetilde{MU}^2(\mathbb{C}P^\infty)$ represented by

$$\xi: \mathbb{C}P^\infty \cong MU(1) \xrightarrow{\text{Id}} MU(1)$$

In fact $\widetilde{MU}^2(\mathbb{C}P^\infty) := [\Sigma^\infty \mathbb{C}P^\infty, \Sigma^2 MU]$ which is seen to be isomorphic to:³

$$\begin{aligned} \widetilde{MU}^2(\mathbb{C}P^\infty) &:= [\Sigma^\infty \mathbb{C}P^\infty, \Sigma^2 MU]_{\text{stable}} \\ &\simeq [\Sigma^\infty \mathbb{C}P^\infty, \Sigma^2 QMU]_{\text{strict}} \\ &\simeq [\mathbb{C}P^\infty, \Omega^\infty \Sigma^2 QMU]_* \\ &= [\mathbb{C}P^\infty, (QMU)_2]_* \end{aligned}$$

Since MU is a CW-spectrum, the $(QMU)_2 \simeq \text{colim}_k \Omega^k MU_{k+2}$, and it still makes sense to consider the class of the identity here (consider $k = 0$ and post-compose the identity with the canonical map to the colimit).

Note that $i^*(x)$ is represented by $S^2 \cong \mathbb{C}P^1 \xrightarrow{i} \mathbb{C}P^\infty \xrightarrow{\xi} MU(1)$ which represents $\iota: \mathbb{S} \rightarrow MU$ and $\Sigma^2\iota: S^2 \rightarrow \Sigma^2 MU$.

We show that the condition that a spectrum E being orientable is sufficient to determine the structure of $\mathbb{E}^*(\mathbb{C}P^\infty)$ as a power series ring.

Proposition 2.0.4. *Let (E, x_E) be an oriented spectrum. Then*

1. $\mathbb{E}^*(\mathbb{C}P^n) \cong \pi_*E[i_n^*(y_E)] / (i_n^*(y_E))^{n+1}$
2. $\mathbb{E}^*(\mathbb{C}P^\infty) \cong \pi_*E[[y_E]]$
3. $\mathbb{E}_*(\mathbb{C}P^n) = \pi_*E\{\alpha_0, \dots, \alpha_n\}$ where α_k is the dual basis element of $(i_n^*(y_E))^k$ under the pairing $\mathbb{E}^*(\mathbb{C}P^n) \otimes \mathbb{E}_*(\mathbb{C}P^n) \rightarrow \pi_*E$.

³see <https://ncatlab.org/nlab/show/Introduction+to+Stable+homotopy+theory+---+1-1> example 14

4. $\mathbb{E}_*(\mathbb{C}P^\infty) = \pi_*E\{\alpha_k \mid k \geq 0\}$ where α_k is the dual basis element of y_E^k under the pairing $\mathbb{E}^*(\mathbb{C}P^\infty) \otimes \mathbb{E}_*(\mathbb{C}P^\infty) \rightarrow \pi_*E$.

5. $\mathbb{E}^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong \pi_*E[[y_1, y_2]]$ where $y_i \in \mathbb{E}^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ is defined by $y_i := p_i^*(y_E)$.

6. $\mathbb{E}_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong \pi_*E\{\alpha_j \otimes \alpha_k \mid j, k \geq 0\}$

Remark 2.0.5. The careful reader should now complain about point (2) of the proposition, since it seems to be in contrast with what we know about *singular* cohomology. In fact we always used the fact that $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[y_H]$ but by point (2) it would be $\mathbb{Z}[[y_H]]$, the ring of *formal series* in the indeterminate x . A priori one has only the cohomology groups, one for each degree. With the cup product (and more generally with a multiplicative structure), we can form the product of two elements. Now we want to gather all of these information in a graded ring. This can be done in two possible ways: The usual convention is to define $H^* = \bigoplus_{n \geq 0} H^n$; however, one could instead consider the product $H^* = \prod_{n \geq 0} H^n$. The second choice is more natural *now* for the following reason: in computing the cohomology ring of each of the projective spaces $\mathbb{C}P^n$, there is no such ambiguity since the grading is finite. So when computing the cohomology of $\mathbb{C}P^\infty$ as a limit over the cohomologies of the $\mathbb{C}P^n$'s we have the *natural option* to take the limit in the category of rings right away, instead of first taking it in groups, and then putting a graded ring structure on the result. Now the limit of polynomial rings for increasing polynomial degree is the formal power series ring, therefore $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[[y_H]]$.

Proof. We will use the standard CW structure of $\mathbb{C}P^n$ and $\mathbb{C}P^\infty$ throughout the proof.

1. Recall that $H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}[i_n^*(y_H)] / (i_n^*(y_H))^{n+1}$, and consider the AHSS for $\mathbb{C}P^1$:

Claim 2.0.6. *The element $i_1^*(y_H) \otimes \iota \in E_2^{2,0}$ represents the orientation class $i_1^*(y_E)$*

Proof. Let us have a look at the AHSS for $\mathbb{C}P^1$:

	\vdots	\vdots	\vdots	\vdots
$H^0(\mathbb{C}P^1) \otimes \mathbb{E}^3(*)$	0	$H^2(\mathbb{C}P^1) \otimes \mathbb{E}^3(*)$	0	\dots
$H^0(\mathbb{C}P^1) \otimes \mathbb{E}^2(*)$	0	$H^2(\mathbb{C}P^1) \otimes \mathbb{E}^2(*)$	0	\dots
$H^0(\mathbb{C}P^1) \otimes \mathbb{E}^1(*)$	0	$H^2(\mathbb{C}P^1) \otimes \mathbb{E}^1(*)$	0	\dots
$H^0(\mathbb{C}P^1) \otimes \mathbb{E}^0(*)$	0	$H^2(\mathbb{C}P^1) \otimes \mathbb{E}^0(*)$	0	\dots
$H^0(\mathbb{C}P^1) \otimes \mathbb{E}^{-1}(*)$	0	$H^2(\mathbb{C}P^1) \otimes \mathbb{E}^{-1}(*)$	0	\dots
	\vdots	\vdots	\vdots	\vdots

Since the edge homomorphism for the AHSS is always surjective, we have that the only possible non zero differentials (the differentials of the 2nd page starting from the zeroth column to the second one) are trivial. This implies that the spectral sequence collapses. Since the second page is generated multiplicatively by $i_1^*(y_H) \otimes \iota \in E_2^{2,0}$ and it is a free graded π_*E -module (i.e. the extension problem is trivial) the isomorphism $\mathbb{E}^2(\mathbb{C}P^1) \xrightarrow{\cong} E_2^{0,2} \oplus E_2^{2,0}$ maps $i_1^*(y_E) \mapsto i_1^*(y_H) \otimes \iota$. \square

Claim 2.0.7. *The AHSS for $\mathbb{C}P^n$ collapses at the second page and we have that $i_n^*(y_H) \otimes \iota \in E_2^{2,0}$ represents the orientation class $i_n^*(y_E)$*

Proof. Let us have a look at the AHSS for $\mathbb{C}P^n$:

$$\begin{array}{cccccc}
& \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
& | & & | & & | & & | & & | \\
H^0(\mathbb{C}P^n) \otimes \mathbb{E}^3(*) & 0 & H^2(\mathbb{C}P^n) \otimes \mathbb{E}^3(*) & 0 & H^4(\mathbb{C}P^n) \otimes \mathbb{E}^3(*) & & & & & \\
H^0(\mathbb{C}P^n) \otimes \mathbb{E}^2(*) & 0 & H^2(\mathbb{C}P^n) \otimes \mathbb{E}^2(*) & 0 & H^4(\mathbb{C}P^n) \otimes \mathbb{E}^2(*) & & & & & \\
H^0(\mathbb{C}P^n) \otimes \mathbb{E}^1(*) & 0 & H^2(\mathbb{C}P^n) \otimes \mathbb{E}^1(*) & 0 & H^4(\mathbb{C}P^n) \otimes \mathbb{E}^1(*) & & & & & \\
H^0(\mathbb{C}P^n) \otimes \mathbb{E}^0(*) & 0 & H^2(\mathbb{C}P^n) \otimes \mathbb{E}^0(*) & 0 & H^4(\mathbb{C}P^n) \otimes \mathbb{E}^0(*) & & & & & \\
\hline
H^0(\mathbb{C}P^n) \otimes \mathbb{E}^{-1}(*) & 0 & H^2(\mathbb{C}P^n) \otimes \mathbb{E}^{-1}(*) & 0 & H^4(\mathbb{C}P^n) \otimes \mathbb{E}^{-1}(*) & & & & & \\
& \vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

Since the AHSS is multiplicative and thanks to the ring structure of the second page, it's enough to show that the element $i_n^*(y_H) \otimes \iota \in E_2^{2,0}$ is an infinite cycle. In fact, one proceed inductively using the fact that any other element in the previous page (i.e. in the second page) is a π_*E -linear combination of powers of $i_n^*(y_H) \otimes \iota$.

Consider the inclusion $i_1^n: \mathbb{C}P^1 \rightarrow \mathbb{C}P^n$. We know that $i_n^*(y_E)$ is sent to $i_1^*(y_E)$ by the fact that $(i_1^n)^*i_n^* = i_1^*$. Now recall that i_1^n induces a map of spectral sequences. Since we know that AHSS for $\mathbb{C}P^n$ converges *a priori* to $\mathbb{E}^*(\mathbb{C}P^n)$ (π_*E is required to be bounded below) there is an element in the stable page $E_\infty^{p,q}$, for some $p, q \in \mathbb{Z}$ which is a representative of $i_n^*(y_E)$. By Claim 2.0.6, we already know that the representative of $i_1^*(y_E)$ lies in $E_\infty^{2,0} \cong F^2\mathbb{E}^2(\mathbb{C}P^1) / F^3\mathbb{E}^2(\mathbb{C}P^1)$. Now suppose that the class of $i_n^*(y_E)$ lives in $F^p\mathbb{E}^{p+q}(\mathbb{C}P^n) / F^{p+1}\mathbb{E}^{p+q}(\mathbb{C}P^n)$, since the inclusion preserves the filtration, it would send our class to an element lying in $F^p\mathbb{E}^{p+q}(\mathbb{C}P^1) / F^{p+1}\mathbb{E}^{p+q}(\mathbb{C}P^1)$, but since we already established that the image of $i_n^*(y_E)$ lies in $F^2\mathbb{E}^2(\mathbb{C}P^1) / F^3\mathbb{E}^2(\mathbb{C}P^1)$, it must be that $p \geq 2$ and $q = 0$ (Since $F^p\mathbb{E}^2(\mathbb{C}P^n) \subseteq F^{p-1}\mathbb{E}^2(\mathbb{C}P^n)$). By definition of filtration, if this element lies in $F^p\mathbb{E}^2(\mathbb{C}P^n) / F^{p+1}\mathbb{E}^2(\mathbb{C}P^n)$ for $p > 2$ in particular its representatives lie in $F^2\mathbb{E}^2(\mathbb{C}P^n)$ meaning that when restricted to $\mathbb{C}P^1$ they are all zero. Since we know that the restriction to $\mathbb{C}P^1$ of the orientation $i_n^*(y_E)$ is a non zero element, it must be $p = 2$ and $q = 0$.

This shows that we have to look for a representative for $i_n^*(y_E)$ in $E_\infty^{2,0}$. Consider the following diagram, where $E_r^{p,q}$ will denote the group in position p, q , page r of the AHSS for $\mathbb{C}P^n$:

$$\begin{array}{ccc}
E_\infty^{2,0} & \xrightarrow{(i_1^n)^*} & E_\infty^{2,0} \\
& \nearrow (i_1^n)^* & \parallel \\
E_2^{2,0} & \xrightarrow[\cong]{(i_1^n)^*} & E_2^{2,0}
\end{array}$$

The lower map is an isomorphism since it is the map induced between the second singular cohomology groups of $\mathbb{C}P^1$ and $\mathbb{C}P^n$. The diagram implies that the unique preimage of the representative of $i_1^*(y_E) \in \mathbb{E}^2(\mathbb{C}P^1)$, which by Claim 2.0.6 is $i_1^*(y_H) \otimes \iota$, has to be $i_n^*(y_H) \otimes \iota$, and therefore it has to be an element of $E_\infty^{2,0}$ i.e. an infinite cycle. This readily implies that the AHSS collapses at the second page, since it is multiplicative: the second page is generated multiplicatively by $i_n^*(y_H) \otimes \iota$, and we just

showed that it is an infinite cycle, by an easy induction we have that the differentials in every page must be zero since they are π_*E -linear derivations. \square

Claim 2.0.8. *As an π_*E -graded module, $E_\infty^{*,*} \cong \mathbb{E}^*(\mathbb{C}P^n)$.*

Proof. This is trivial, since using Claim 2.0.7 the stable page is the second page, and it is clearly a free π_*E -graded module. Therefore the extension problem is trivial as remarked in the crash course on spectral sequences, and we have the claim. \square

Claim 2.0.9. *As a ring we have $E_\infty^{*,*} \cong \mathbb{E}^*(\mathbb{C}P^n)$.*

Proof. We will denote with \smile the product on $\mathbb{E}^*(\mathbb{C}P^n)$ and with \bullet the one on the stable page. Using Claim 2.0.8, it's enough to prove that the multiplicative structure one has (by definition) on $\mathbb{E}^*(\mathbb{C}P^n)$, coincide with the one of $E_\infty^{*,*}$ seen as $\pi_\bullet E[i_n^* y_E] / (i_n^* y_E)^{n+1}$. To this end, since $\mathbb{E}^*(\mathbb{C}P^n) \cong \pi_\bullet E[i_n^* y_E, (i_n^* y_E)^2, \dots, (i_n^* y_E)^n]$ it suffices to prove that $(i_n^* y_E)^m \smile (i_n^* y_E)^k = (i_n^* y_E)^{m+k \pmod{n+1}}$. Using associativity and induction it's enough to prove that $(i_n^* y_E) \smile (i_n^* y_E)^m = (i_n^* y_E)^{m+1 \pmod{n+1}}$. We know that $i_n^* y_E \in E_\infty^{2,0}$ and that $(i_n^* y_E)^m \in E_\infty^{2m,0}$, and on the stable page $(i_n^* y_E) \smile (i_n^* y_E)^m = (i_n^* y_E)^{m+1 \pmod{n+1}}$. By the compatibility of the product structure of the stable page and on the cohomology ring, this relation must hold on $\mathbb{E}^*(\mathbb{C}P^n)$. \square

2. For $m < n$ let $i_{m,n}: \mathbb{C}P^m \rightarrow \mathbb{C}P^n$ denote the inclusion map. Since the inverse system $i_{m,n}^*: \mathbb{E}^*(\mathbb{C}P^n) \rightarrow \mathbb{E}^*(\mathbb{C}P^m)$ satisfies the Mittag-Leffler condition

$$\mathbb{E}^*(\mathbb{C}P^\infty) \cong \varinjlim \mathbb{E}^*(\mathbb{C}P^n) \cong \varinjlim \pi_\bullet E[i_n^* y_E] / (i_n^* y_E)^{n+1} \cong \pi_\bullet E[[y_E]]$$

3. Consider the homological AHSS for $\mathbb{E}_*(\mathbb{C}P^n)$, using UCT we have:

$$E_{s,t}^2 = H_s(\mathbb{C}P^n) \otimes \mathbb{E}_t(*) \Rightarrow \mathbb{E}_{s+t}(\mathbb{C}P^n)$$

Recall that, as a \mathbb{Z} -module we have $H_*(\mathbb{C}P^n) \cong \mathbb{Z}\{a_0, \dots, a_n\}$ where a_k is the dual of $(i_n^*(y_E))^k$ under the identification $H_*(\mathbb{C}P^n) \cong \text{hom}(H^*(\mathbb{C}P^n); \mathbb{Z})$. Recall that we have a pairing for the AHSSes for $\mathbb{E}^*(\mathbb{C}P^n)$ and $\mathbb{E}_*(\mathbb{C}P^n)$.

Claim 2.0.10. *On the stable page the pairing $E_{k,0}^\infty \otimes E_\infty^{k,-s} \rightarrow \mathbb{E}_s(*)$ is non-degenerate, i.e.*

$$\langle x, y \rangle = 0 \quad \forall x \Rightarrow y = 0$$

and the AHSS for $\mathbb{E}_*(\mathbb{C}P^n)$ collapses

Proof. We merged these two statements into one claim since we will prove inductively one of them assuming the previous step of the other one and so on.

On the second page, by Prop 4.0.6 we have that the pairing is non degenerate. In fact by our choice of generator of the homology group

$$\langle (i_n^*(y_E))^k \otimes \iota, a_k \otimes b \rangle = b$$

and clearly $\iota \in \pi_0 E$. This means that

$$0 \stackrel{(1)}{=} \langle d^2(i_n^*(y_E))^k \otimes \iota, a_k \otimes b \rangle \stackrel{(2)}{=} \langle (i_n^*(y_E))^k \otimes \iota, d_2 a_k \otimes b \rangle \stackrel{(3)}{\Rightarrow} d_2 a_k \otimes b = 0$$

where (1) is due to $d^2 = 0$, (2) is by the properties of the pairing and (3) since by linearity of the differentials and of the pairing it's enough to test $d_2 a_k \otimes b$ against $(i_n^*(y_E))^k \otimes \iota$ alone in order to invoke the non-degeneracy property. So we just proved that $d_2 = 0$. Now we can proceed inductively since $E_2^{k,0} = E_r^{k,0} = E_\infty^{k,0}$ and the inductive hypothesis $E_{k,-s}^2 = E_{k,-s}^n$ to show that the pairing is non-degenerate (since it coincides with the one on the second page) and $d_n = 0$ which implies $E_{n+1}^{k,-s} = E_n^{k,-s}$ and we can continue for higher indices. Since we are dealing with a convergent spectral sequence we can conclude that

$$\langle \cdot, \cdot \rangle: E_{k,0}^\infty \otimes E_\infty^{k,-s} \rightarrow \mathbb{E}_s(*)$$

is non degenerate and that the homological AHSS collapses at the second page. \square

For the same reason as in the proof of the first point, we can identify $E_\infty^{*,*}$ with $\mathbb{E}_*(\mathbb{C}P^n)$. Using the fact that the pairing between the stable pages is induced by the one between homology and cohomology we have a non degenerate pairing

$$\mathbb{E}^*(\mathbb{C}P^n) \otimes \mathbb{E}_*(\mathbb{C}P^n) \rightarrow \pi_* E$$

Claim 2.0.11. *The pairing $\mathbb{E}^*(\mathbb{C}P^n) \otimes \mathbb{E}_*(\mathbb{C}P^n) \rightarrow \pi_* E$ induces an isomorphism $\mathbb{E}_*(\mathbb{C}P^n) \cong \text{hom}_{\pi_* E}(\mathbb{E}^*(\mathbb{C}P^n), \pi_* E)$.*

Proof. We just showed that the pairing coincides with the one on the stable page, and the pairing on the infinity page is the one on the second page since both AHSSes collapse. On the second page it's clear that the pairing is unimodular since it's unimodular by construction the pairing $H^*(\mathbb{C}P^n) \otimes H_*(\mathbb{C}P^n) \rightarrow \mathbb{Z}$. The conclusion follows at once. \square

Using Claim 2.0.11, we set $\alpha_k = (i_n^*(y_E)^k)^*$, which by combining Claim 2.0.11 and Claim 2.0.6 is represented by a_k , and thus $\mathbb{E}_*(\mathbb{C}P^n) \cong \pi_* \{\alpha_0, \dots, \alpha_n\}$.

4. We need the following claim

Claim 2.0.12. $\mathbb{E}_*(\mathbb{C}P^\infty) \cong \text{colim}_n \mathbb{E}_*(\mathbb{C}P^n)$

Proof. Apply Proposition 4.2.2 page 121 in [1] to $X = \mathbb{C}P^\infty \coprod *$ and $X^n = \mathbb{C}P^n \coprod *$ \square

Since $\mathbb{C}P^\infty = \bigcup_{n \geq 1} \mathbb{C}P^n$ we have

$$\mathbb{E}_*(\mathbb{C}P^\infty) \cong \text{colim}_n \mathbb{E}_*(\mathbb{C}P^n) \cong \text{colim}_n \pi_* E \{\alpha_0, \dots, \alpha_n\} \cong \pi_* E \{\alpha_0, \dots, \alpha_n, \dots\}$$

5. Let $p_i: \mathbb{C}P^\infty \times \mathbb{C}P^\infty \rightarrow \mathbb{C}P^\infty$ for $i = 1, 2$ denote the two projection maps. Define $y_i \in \mathbb{E}^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ by $y_i := p_i^*(y_E)$. Let $j_{mn}: \mathbb{C}P^m \times \mathbb{C}P^n \rightarrow \mathbb{C}P^\infty \times \mathbb{C}P^\infty$ denote the inclusion map. Since

$$H^*(\mathbb{C}P^m \times \mathbb{C}P^n) \cong H^*(\mathbb{C}P^m) \otimes H^*(\mathbb{C}P^n) \cong \mathbb{Z}[y_1, y_2] / (y_1^{m+1}, y_2^{n+1})$$

where the first isomorphism is induced by the cross product. Recall that $a \times b = \pi_1^* a \smile \pi_2^* b$ (where π_i are the obvious projections) and therefore by naturality together with multiplicativity of the AHSS, one can show inductively that the AHSS for the product $\mathbb{E}^*(\mathbb{C}P^m \times \mathbb{C}P^n)$ collapses at the second page. We can then apply *verbatim* the proof of the first point to conclude

$$\mathbb{E}^*(\mathbb{C}P^m \times \mathbb{C}P^n) \cong \pi_* E [j_{mn}^*(y_1), j_{mn}^*(y_2)] / (j_{mn}^*(y_1)^{m+1}, j_{mn}^*(y_2)^{n+1})$$

To conclude, as in the proof of the second point

$$\begin{aligned} \mathbb{E}^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) &\cong \text{colim}_{m,n} \mathbb{E}^*(\mathbb{C}P^m \times \mathbb{C}P^n) \\ &\cong \text{colim}_{m,n} \pi_* E [j_{mn}^*(y_1), j_{mn}^*(y_2)] / (j_{mn}^*(y_1)^{m+1}, j_{mn}^*(y_2)^{n+1}) \\ &\cong \pi_* E [[y_1, y_2]] \end{aligned}$$

6. The proof of the this last point is left as an exercise. \square

Recall this result:

Proposition 2.0.13. *There is a map $\mu_n: \prod_{i=1}^n BU(1) \rightarrow BU(n)$ which defines in homology an associative and commutative product*

$$(\mu_n)_*: \bigotimes_{i=1}^n H_*(BU(1)) \rightarrow H_*(BU(n))$$

We will denote by $\mu_{t,n}$ the composition $\mu_{t,n}: \prod_{i=1}^t BU(1) \rightarrow BU(t) \rightarrow BU(n)$.

- There are $c_k \in H^{2k}(BU(n))$ for $1 \leq k \leq n$ called universal Chern Classes such that

$$H^*(BU(n)) \cong \mathbb{Z}\{c_1, \dots, c_n\}$$

- Let $a_{k_i} \in H_{2k}(BU(1))$ defined as $a_k = (c_i^k)^*$. We still denote a_k the element $(\mu_{1,n})_*(a_k) \in H_{2k}(BU(n))$ with a little abuse of notation. Define

$$a_{k_1} \cdots a_{k_t} := (\mu_{t,n})_*(a_{k_1} \otimes \cdots \otimes a_{k_t})$$

Then $H_*(BU(n))$ is the free abelian group with basis the set of all $a_{k_0} \cdots a_{k_t}$ for $0 \leq t \leq n$.

Proof. See Theorem 2.3.1 page 39 and Proposition 2.4.1 page 48 on [1] □

Using induction on $n \geq 1$ we use various pairings of the AHSS to compute $\mathbb{E}_*(BU(n))$ and $\mathbb{E}^*(BU(n))$ for an oriented spectrum E .

Proposition 2.0.14. *Let E be an oriented spectrum*

1. The map $\mu_n: \prod_{i=1}^n BU(1) \rightarrow BU(n)$ induces a map

$$\bigotimes_n \mathbb{E}_*(BU(1)) \rightarrow \mathbb{E}_*(BU(n))$$

Using this product, $\mathbb{E}_*(BU(n))$ is the free π_*E -module with basis

$$\{\alpha_{k_1} \cdots \alpha_{k_t} \mid 1 \leq k_1 \leq \cdots \leq k_t \text{ and } t \leq n\}$$

2. There are classes $cf_k \in \mathbb{E}^{2k}(BU(n))$ for $1 \leq k \leq n$ called Conner-Floyd classes such that

$$\mathbb{E}^*(BU(n)) \cong \pi_*[[cf_1, \dots, cf_n]]$$

3. $\mathbb{E}^*(BU) \cong \pi_*E[[cf_1, \dots, cf_n, \dots]]$

4. $\mathbb{E}_*(BU) = \pi_*E[\alpha_1, \dots, \alpha_n, \dots]$

Proof. We will make use of induction on $n \geq 1$ since $BU(1) \simeq \mathbb{C}P^\infty$ and extensive use of naturality of AHSS.

1. The base step is done in the previous prop. Assume $n \geq 2$ and consider the AHSS

$$E_{k,t}^2 = H_k(BU(n)) \otimes \pi_t E \Rightarrow \mathbb{E}_*(BU(n))$$

We know that $H_*(BU(n))$ is the free abelian group with basis

$$\{a_{k_1} \cdots a_{k_t} \mid t \leq n\}$$

We want to prove that AHSS collapses at the second page. By naturality and induction, $d_r(a_{k_1} \cdots a_{k_t} \otimes \eta) = 0$ if $t < n$ since such an element comes from the AHSS of $BU(n-1)$ and we know that it collapses at the second page. So we can assume that $t = n$. We want to show that $d_r = 0$ for all $r \geq 2$ and by linearity of the differentials it's enough to show it on elements of the form $a_{k_1} \cdots a_{k_n} \otimes \iota$. Consider the map

$$\mu_n: \prod_{i=1}^n BU(1) \rightarrow BU(n)$$

$$\begin{aligned} d_r(a_{k_1} \cdots a_{k_n} \otimes \iota) &= d_r((\mu_n)_*(a_{k_1} \otimes \cdots \otimes a_{k_n}) \otimes \iota) \\ &= (\mu_n)_* d_r((a_{k_1} \otimes \cdots \otimes a_{k_n}) \otimes \iota) \\ &= (\mu_n)_* d_r((a_{k_1} \times \cdots \times a_{k_n}) \otimes \iota) \end{aligned}$$

where the last passage is due to the Künneth isomorphism in singular homology. Since we can express cross product via cup product and induced maps, naturality and multiplicativity show that $a_{k_1} \cdots a_{k_t}$ is an infinite cycle which represents $\alpha_{k_1} \cdots \alpha_{k_t}$. Therefore the sequence collapses and since the stable page is a free π_*E -module, $\mathbb{E}_*(BU(n)) \cong E_{*,*}^\infty$ form a π_*E -basis for $\mathbb{E}_*(BU(n))$.

2. Fix a positive even integer q . Consider the pairing of the AHSSes

$$E_2^{k,t} = H^k(BU(n)^q) \otimes E^t \Rightarrow \mathbb{E}^*(BU(n)^q)$$

and

$$E_{k,t}^2 = H_k(BU(n)^q) \otimes E_t \Rightarrow \mathbb{E}_*(BU(n)^q)$$

(The pairing exists for finite CW, that's why we have to fix an even number q) Note that by Prop. 2.0.13 $H_*(BU(n)^q)$ has a basis consisting of the set of all $a_{k_1} \cdots a_{k_t}$ of degree at most q . Thus, the homological spectral sequence listed above is a subspectral sequence of the one seen in the point before and collapses. Therefore,

$$\mathbb{E}_*(BU(n)^q) = \pi_* E \{ \alpha_{k_1} \cdots \alpha_{k_t} \mid t \leq n \text{ and } k_1 + \cdots + k_t \leq q/2 \}$$

Recall that $H^*(BU(n)) = \mathbb{Z}[c_1, \dots, c_n]$, where $c_k = (a_1^k)^*$ under the identification of $H^*(BU(n))$ with $\text{hom}(H_*(BU(n)), \mathbb{Z})$. Therefore, $H^*(BU(n)^q)$ is the subgroup of $\mathbb{Z}[c_1, \dots, c_n]$ with basis all monomials in the c_k of degree at most q . Consider the pairing of the two spectral sequences wrote above. It's easy to see that on the second page such pairing is uni-modular and by the same reasoning of the previous proposition we have a uni-modular pairing $\mathbb{E}^*(BU(n)^q) \otimes \mathbb{E}_*(BU(n)^q) \rightarrow \pi_* E$ which induces an isomorphism

$$\mathbb{E}^*(BU(n)^q) \cong \text{hom}_{\pi_* E}(\mathbb{E}_*(BU(n)^q), \pi_* E)$$

Let $cf_k = (\alpha_1^k)^*$ which projects to $c_k \in E_\infty^{2k,0}$. Then $\mathbb{E}^*(BU(n)^q)$ is the free $\pi_* E$ -module with basis all monomials in the cf_1, \dots, cf_n of degree at most q . Since the inverse system of the $\mathbb{E}^*(BU(n)^q)$ satisfies the Mittag-Leffler condition,

$$\mathbb{E}^*(BU(n)) = \varinjlim_q \mathbb{E}^*(BU(n)^q) = \pi_* E[[cf_1, \dots, cf_n]]$$

3. Just verify the Mittag-Leffler condition and conclude

4. homology preserve direct limits.

□

3 A brief recall of (B, f) -structures

Recall the following: let $BO(n)$ be the infinite dimensional Grassmannian of n -planes, i.e. $\text{Gr}(n, \mathbb{R}^\infty)$. It's well-known that $BO(n)$ is the classifying space for the n -dimensional real vector bundle. We have a universal $O(n)$ -principal bundle $EO(n) \rightarrow BO(n)$, where $EO(n)$ is the so called Stiefel manifold $V(n, \mathbb{R}^\infty)$, seen as a colimit of finite dimensional Stiefel manifolds $V(n, \mathbb{R}^k)$, for $n \leq k$. Recall that $V(n, \mathbb{R}^k)$ is $n-1$ connected.

Definition 3.0.1. A (B, f) structure \mathfrak{B} is a collection of pointed spaces B_n and strictly commutative diagrams

$$\begin{array}{ccc} B_n & \xrightarrow{f_n} & BO(n) \\ \downarrow g_n & & \downarrow Bi_n \\ B_{n+1} & \xrightarrow{f_{n+1}} & BO(n+1) \end{array}$$

where the maps f_n are required to be based fibrations. We require additionally that $B_0 = *$.

As a shorthand for the iterated composition of the g_n 's, we will denote with $g_{n,m} := g_{m-1} \circ g_{m-2} \circ \cdots \circ g_n$. We will denote with O the *trivial* (B, f) -structure with $B_n = BO(n)$ and $f_n = \text{Id}$. Other well-known (B, f) -structures which we will use often are EO and U , the framed and stable almost complex structure respectively. Their precise definition can be found on [1] at page 14 – 15.

Definition 3.0.2. A multiplicative (B, f) -structure \mathfrak{B} also has natural based maps

$$\mu_{n,m}^{\mathfrak{B}}: B_n \times B_m \rightarrow B_{n+m}$$

for all $n, m \in \mathbb{N}$, such that the following diagrams commute:

- **Naturality**

$$\begin{array}{ccc} B_n \times B_m & \xrightarrow{\mu_{n,m}^{\mathfrak{B}}} & B_{n+m} \\ f_n \times f_m \downarrow & & \downarrow f_{n+m} \\ BO(n) \times BO(m) & \xrightarrow{\mu_{n,m}^O} & BO(n+m) \end{array}$$

- **Associativity**

$$\begin{array}{ccc} B_n \times B_m \times B_k & \xrightarrow{\mu_{n,m}^{\mathfrak{B}} \times 1} & B_{n+m} \times B_k \\ 1 \times \mu_{m,k}^{\mathfrak{B}} \downarrow & & \downarrow \mu_{n+m,k}^{\mathfrak{B}} \\ B_n \times B_{m+k} & \xrightarrow{\mu_{n,m+k}^{\mathfrak{B}}} & B_{n+m+k} \end{array}$$

- **Compatibility and Unitality** We require that the product is compatible with the g_n 's in the following ways.

$$\begin{array}{ccccc} B_n & \xrightarrow{1 \times *}& B_n \times B_m & \xleftarrow{* \times 1}& B_m \\ & \searrow g_{n,m+n} & \downarrow \mu_{n,m}^{\mathfrak{B}} & \swarrow g_{m,n+m} & \\ & & B_{n+m} & & \end{array}$$

Specializing to $B_0 = *$ and $g_{n,n} = \text{Id}$ we get that the product $\mu_{n,m}^{\mathfrak{B}}$ has a unit, namely $*$.

$$\begin{array}{ccc} B_n \times B_m & \xrightarrow{1 \times g_{m,m+k}} & B_n \times B_{m+k} \\ \mu_{n,m}^{\mathfrak{B}} \downarrow & & \downarrow \mu_{n,m+k}^{\mathfrak{B}} \\ B_{n+m} & \xrightarrow{g_{n+m,n+m+k}} & B_{n+m+k} \\ \mu_{n,m}^{\mathfrak{B}} \uparrow & & \uparrow \mu_{n+k,m}^{\mathfrak{B}} \\ B_n \times B_m & \xrightarrow{g_{n,n+k} \times 1} & B_{n+k} \times B_m \end{array}$$

Definition 3.0.3. Let \mathfrak{B} and \mathfrak{B}' be two (B, f) -structures. A map of (B, f) -structures $h: \mathfrak{B} \rightarrow \mathfrak{B}'$ is a collection of maps h_n such that the following diagram commutes, for every $n \in \mathbb{N}$.

$$\begin{array}{ccc} B_n & \xrightarrow{h_n} & B'_n \\ \downarrow g_n & & \downarrow g'_n \\ B_{n+1} & \xrightarrow{h_{n+1}} & B'_{n+1} \end{array}$$

If \mathfrak{B} and \mathfrak{B}' are multiplicative (B, f) -structures, then h is a map of multiplicative (B, f) -structures if, in addition, for all indexes $n, m \in \mathbb{N}$, the following diagram commutes.

$$\begin{array}{ccc}
B_n \times B_m & \xrightarrow{\mu_{n,m}^{\mathfrak{B}}} & B_{n+m} \\
h_n \times h_m \downarrow & & \downarrow h_{m+n} \\
B'_n \times B'_m & \xrightarrow{\mu_{n,m}^{\mathfrak{B}'}} & B'_{m+n}
\end{array}$$

Remark 3.0.4. The first example of a (multiplicative) (B, f) structure is the trivial one, denoted by BO , where the fibrations are simply the respective identities. Another (B, f) structure is the one originating by the bundle $EO(n) \rightarrow BO(n)$ (interpreted as a vector bundle)

$$\begin{array}{ccc}
EO(n) & \xrightarrow{Ei_n} & EO(n+1) \\
\downarrow p_n & & \downarrow p_{n+1} \\
BO(n) & \xrightarrow{Bi_n} & BO(n+1)
\end{array}$$

Remark 3.0.5. Please note that if \mathfrak{B} is a (multiplicative) (B, f) -structure, we require that there is a map of (multiplicative) (B, f) -structure $h: EO \rightarrow \mathfrak{B}$.

Definition 3.0.6. A (B, f) -structure \mathfrak{B} on a smooth manifold M is a pair $(h, \tilde{\nu})$ such that $h: M \rightarrow \mathbb{R}^{n+k}$ is an embedding with normal bundle $N \rightarrow M$ classified by ν and $\tilde{\nu}$ is a lifting $\tilde{\nu}: M \rightarrow B_k$ of ν :

$$\begin{array}{ccc}
& & B_k \\
& \nearrow \tilde{\nu} & \downarrow f_k \\
M & \xrightarrow{\nu} & BO(k)
\end{array}$$

Proposition 3.0.7. An EO -structure (e, g) on a smooth manifold M^n corresponds to a framing of the normal bundle ν (associated to e) of M^n

Proof. It's easy to observe that the lifting of the classifying map g to $EO(k)$ for some k corresponds to a continuous association for each point $m \in M$ of a k -frame for the normal bundle (which is assumed to be k -dimensional. More formally (and less geometric) a map $\nu: M^n \rightarrow BO(k)$ factor through a contractible space $EO(k)$ if and only if it's null-homotopic, i.e. if and only if the bundle classified by ν is trivial. \square

Example 3.0.8. A manifold with BU -structure is also called a stable almost complex structure, since up to stabilization its normal bundle admits an almost complex structure. Of course, complex manifold are stably almost complex.

There is another interesting example that we want to consider.

Example 3.0.9. Recall that the quaternions \mathbb{H} are the non commutative division algebra defined by

$$\mathbb{H} = \mathbb{R}\langle 1, i, j, k \rangle$$

where $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$. We define the group $Sp(n) = GL_{\mathbb{H}}(\mathbb{H}^n) \cap U(2n)$ called the *compact symplectic group*. Being a subgroup of $U(2n) \subset O(4n)$ we can build fibrations $BSp(n) \rightarrow BO(4n)$. Manifold with a Sp -structure are called stable quaternionic manifolds, i.e. manifold whose normal bundle (up to stabilization) admits a quaternionic structure.

We define a the equivalence relation of bordism on the set of manifolds with (B, f) -structure \mathfrak{B} . Disjoint union makes the set of equivalence classes $\Omega_*^{\mathfrak{B}}$ into a graded group and if \mathfrak{B} is a multiplicative structure then the Cartesian product makes $\Omega_*^{\mathfrak{B}}$ into a graded ring.

We define the negative of a manifold (M^n, e, g) as the manifold at level 1 of the manifold with \mathfrak{B} -structure $(M \times I, e \times i, g \times fr)$ where

$$\begin{aligned}
e \times i: M \times I &\rightarrow \mathbb{R}^{n+k} \oplus \mathbb{R}^2 \\
(m, t) &\mapsto (e(m), \cos(\pi t) \cdot e_1 + \sin(\pi t) \cdot e_2) \\
g \times fr: M \times I &\rightarrow B_k \times EO(1) \subseteq B_{k+1} \\
(m, t) &\mapsto (g(m), \cos(\pi t)e_1 + \sin(\pi t)e_2)
\end{aligned}$$

where we used the fact that we require the existence of a map $EO \rightarrow \mathfrak{B}$.

Definition 3.0.10. Two closed manifolds with \mathfrak{B} -structure (M^n, e, g) and (N^n, f, h) are bordant if there is a manifold with \mathfrak{B} -structure (W^{n+1}, E, G) such that

$$\partial(W^{n+1}, E, G) = (M^n, e, g) \sqcup -(N^n, f, h)$$

Lemma 3.0.11. *Bordism is an equivalence relation*

Proof. This is Lemma 1.5.2 in [1]. □

Proposition 3.0.12. *Let \mathfrak{B} be a (B, f) -structure.*

- *The set $\Omega_*^{\mathfrak{B}}$ is a graded abelian group under the operation of disjoint union*
- *If \mathfrak{B} is a multiplicative (B, f) -structure then $\Omega_*^{\mathfrak{B}}$ is a graded ring with the operation of multiplication given by the Cartesian product.*

Proof. This is Proposition 1.5.3 in [1]. □

The construction of the Thom Spectra MU and the Thom-Pontrjaing Theorem

Now let \mathfrak{B} be a (B, f) -structure, and let ω_n denote the universal $O(n)$ -bundle over $BO(n)$. Recall that we have the following commutative diagram:

$$\begin{array}{ccc} B_n & \xrightarrow{g_n} & B_{n+1} \\ \downarrow f_n & & \downarrow f_{n+1} \\ BO(n) & \xrightarrow{Bi_n} & BO(n+1) \end{array}$$

Over B_n we have the bundle $\gamma_n = f_n^* \omega_n$ and

$$g_n^* \gamma_{n+1} = g_n^* f_{n+1}^* \omega_{n+1} \simeq f_n^* (Bi_n)^* \omega_{n+1} \simeq f_n^* (\omega_n \oplus \epsilon^1) \simeq f_n^* \omega_n \oplus \epsilon^1 = \gamma_n \oplus \epsilon^1$$

where the only non-trivial check is $(Bi_n)^* \omega_{n+1} = \omega_n \oplus \epsilon^1$.

Thus g_n induces a bundle map

$$\gamma_n \oplus \epsilon^1 \rightarrow \gamma_{n+1}$$

and hence a map

$$M(g_n): \Sigma M(\gamma_n) \rightarrow M(\gamma_{n+1})$$

of Thom complexes.

Definition 3.0.13. We define the *Thom Spectrum* $M\mathfrak{B}$ to be the following spectrum: as objects we have $(M\mathfrak{B})_n := M(\gamma_n)$ and as structure maps we take $M(g_n): \Sigma(M\mathfrak{B})_n \rightarrow (M\mathfrak{B})_{n+1}$

In particular $MO(n) := \text{Th}(\gamma_n)$. In order to define MU , we consider the following (B, f) -structure

$$\begin{array}{ccc} BU(n) & \xrightarrow{i_n} & BO(2n) \\ \downarrow \text{Id} & & \downarrow Bi_n \\ BU(n) & \xrightarrow{Bi_n \circ i_n} & BO(2n+1) \end{array}$$

and a (B, f) -structure for a manifold represents exactly a stable almost complex structure. Pulling back via the inclusion i_n the universal bundle over $BO(2n)$ gives us the universal bundle over $BU(n)$ (call it $\omega_n: EU(n) \rightarrow BU(n)$). We define the spectrum MU as follows: $(MU)_{2n} = MU(n)$ and $(MU)_{2n+1} = \Sigma MU(n)$. Structure maps are the maps induced by the classifying map $r_n: BU(n) \rightarrow BU(n+1)$.

Lemma 3.0.14. $MO(1) \simeq \mathbb{R}P^\infty$ and $MU(1) \simeq \mathbb{C}P^\infty$.

Proof. This is Lemma 2.6.5 page 61 in [1]. Let

$$\pi: EO(1) \times_{O(1)} \mathbb{R} \rightarrow BO(1)$$

the associated vector bundle to the principal bundle γ_1 . By construction its Thom space is $MO(1)$. Recall that $BO(1) = \mathbb{R}P^\infty$, and let π_n denote the pullback of π to $\mathbb{R}P^n$. Then the sphere bundle of π_n is the canonical S^0 -bundle $S^n \rightarrow \mathbb{R}P^n$, Thus

$$S(\pi) = \lim_{n \rightarrow \infty} S^n = S^\infty$$

which is contractible. Since the image of the zero section of π is a strong deformation retract of $D(\pi)$, the zero section from $BO(1)$ to $D(\pi) / S(\pi) = MO(1)$ is a homotopy equivalence. Analogously for $MU(1) \simeq \mathbb{C}P^\infty$ \square

Remark 3.0.15. It is worth mentioning that if we have a multiplicative (B, f) -structure \mathfrak{B} , then we obtain a ring spectrum $M\mathfrak{B}$, whose product is the one induced by the product we have on \mathfrak{B} , and unit $i: \mathbb{S} \rightarrow M\mathfrak{B}$ induced by inclusion of the fibre $i_0: S^0 \rightarrow M\mathfrak{B}_0$ over the basepoint.

We can now state the celebrated Pontrjagin-Thom isomorphism in this more general setting

Theorem 3.0.16. *Let \mathfrak{B} be a (multiplicative) (B, f) structure. We have an isomorphism of (rings) graded abelian group*

$$\begin{aligned} \xi: \Omega_*^{\mathfrak{B}} &\rightarrow \pi_*(M\mathfrak{B}) \\ [M, e, g] &\mapsto Mg \circ c \end{aligned}$$

where $c: S^{n+k} \rightarrow M(\nu)$ is the Pontrjagin-Thom collapse map, where we assumed $e: M \rightarrow \mathbb{R}^{n+k}$ and we denoted with $M(\nu)$ the Thom space of the normal bundle ν of (M, e) .

4 A Crash Course in Spectral Sequences

What is a Spectral Sequence

We will list here the main properties of a first quadrant spectral sequence, clearly this will be only a brief introduction but the aim is to motivate certain passages and reasoning that will be used in the following sections:

Definition 4.0.1. A cohomology spectral sequence (starting at the page E_a) in an Abelian Category \mathcal{A} consists of the following data:

1. A family $\{E_r^{pq}\}$ of objects of \mathcal{A} defined for all integers p, q and $r \geq a$
2. Maps $d_r^{pq}: E_r^{pq} \rightarrow E_r^{p+r, q-r+1}$ that are differentials in the sense that $d_r d_r = 0$
3. Isomorphism between E_{r+1}^{pq} and the cohomology of (E_r^{**}, d_r) at the spot E_r^{pq} :

$$E_{r+1}^{pq} \simeq \ker(d_r^{pq}) / \text{Im}(d_r^{p-r, q+r-1})$$

Example 4.0.2. A *first quadrant spectral sequence* is one with $E_r^{pq} = 0$ unless $p, q \geq 0$, that is, the point (p, q) lies in the first quadrant of the plane. (If this condition holds for $r = a$, it clearly holds for all r). If we fix p and q , then $E_r^{pq} = E_{r+1}^{pq}$ for all large r (more specifically $r > \max\{p, q + 1\}$), because the d_r landing in (p, q) spot come from the second quadrant (i.e is 0), and the d_r leaving E_r^{pq} land in the fourth quadrant (i.e. is 0). We write E_∞^{pq} for this stable value of E_r^{pq} .

Definition 4.0.3 (Bounded Convergence). We say that a bounded spectral sequence converges to H^* if we are given a family of objects H^n of \mathcal{A} , each having a *finite* decreasing filtration

$$0 = F^s H^n \subseteq \dots \subseteq F^{p+1} H^n \subseteq F^p H^n \subseteq F^{p-1} H^n \subseteq \dots \subseteq F^t H^n = H^n$$

and we are given isomorphisms $E_\infty^{pq} \simeq F^p H^{p+q} / F^{p+1} H^{p+q}$.

In almost all of the applications of Spectral Sequences in this seminar, H^n will be a filtered module for any $n \in \mathbb{N}$. In this setting, we call that the graded module

$$\text{gr}H^n := \bigoplus_m F^m H^n / F^{m+1} H^n$$

the associated graded module.

This suggests the first big limit of spectral sequences: they give to us information about the associated graded module (see the definition of the infinity page) and in order to retrieve H^n from $\text{gr}H^n$ we have to solve the (highly non trivial) *extension problem*. In order to explain what is this problem, consider the stable page of a first quadrant Spectral Sequence:

$$\begin{array}{c|ccccc} & \vdots & \vdots & \vdots & \vdots & \vdots \\ & E_\infty^{0,3} & E_\infty^{1,3} & E_\infty^{2,3} & E_\infty^{3,3} & E_\infty^{4,3} \\ & E_\infty^{0,2} & E_\infty^{1,2} & E_\infty^{2,2} & E_\infty^{3,2} & E_\infty^{4,2} \\ & E_\infty^{0,1} & E_\infty^{1,1} & E_\infty^{2,1} & E_\infty^{3,1} & E_\infty^{4,1} \\ & E_\infty^{0,0} & E_\infty^{1,0} & E_\infty^{2,0} & E_\infty^{3,0} & E_\infty^{4,0} \\ \hline & 0 & 0 & 0 & 0 & 0 \\ & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

We know that $E_\infty^{0,0} = F_0 H_0 / F_1 H_0$. By the fact that all the elements on the same diagonal are 0, we have $0 \cong E_\infty^{n,-n} \cong F^n H^0 / F^{n+1} H^0$ hence for $n > 0$ $F^n H^0 = F^{n+1} H^0$. Therefore after climbing this ladder of equalities we get inductively $F^1 H^0 = 0$. An analogous reasoning using the second quadrant this time shows that $F^0 H^0 = H^0$. Therefore by the very definition of convergence

$$E_\infty^{0,0} \stackrel{\text{def.}}{\cong} F^0 H^0 / F^1 H^0 \cong H^0.$$

Let's have a look at the first diagonal, again by definition of convergence we have the following isomorphism:

$$E_\infty^{1,0} \cong F^1 H^1 / F^2 H^1 \quad E_\infty^{0,1} \cong F^0 H^1 / F^1 H^1.$$

Using the same reasoning of the preceding case (i.e. we have 0 on the second and fourth quadrants) we can conclude that $F^0 H^1 \cong H^1$ and $F^2 H^1 \cong 0$, therefore $E_\infty^{0,1} \cong H^1 / E_\infty^{1,0}$. Equivalently we have the following s.e.s.

$$0 \rightarrow E_\infty^{1,0} \rightarrow H^1 \rightarrow E_\infty^{0,1} \rightarrow 0$$

Now we proceed as before for the second diagonal and we have

$$E_\infty^{2,0} \cong F^2 H^2 / F^3 H^2 \quad E_\infty^{1,1} \cong F^1 H^2 / F^2 H^2 \quad E_\infty^{0,2} \cong F^0 H^2 / F^1 H^2.$$

which gives us the following two short exact sequences

$$\begin{aligned} 0 \rightarrow E_\infty^{2,0} \rightarrow F^1 H^2 \rightarrow E_\infty^{1,1} \rightarrow 0 \\ 0 \rightarrow F^1 H^2 \rightarrow H^2 \rightarrow E_\infty^{0,2} \rightarrow 0 \end{aligned}$$

In general once you reach the stable page, just solve *consecutive* extension problems starting from $E_\infty^{p,0}$, and the last extension will provide you the right result. We can sum up together all this information in the following short exact sequence of graded modules:

$$0 \rightarrow F^{\bullet+1} H^{p+\bullet} \rightarrow F^\bullet H^{p+\bullet} \rightarrow E_\infty^{p,\bullet} \rightarrow 0$$

This will turn out to be important later. Recall that if for some reason we know that the s.e.s. are split exact then we can easily retrieve H^* from the extension problem since inductively we get

$$H^k \cong \bigoplus_{p+q=k, p, q \geq 0} E_\infty^{pq}$$

Additional Structure on a Spectral Sequence

We will list here a list of additional feature of a spectral sequence which can come in handy for extrapolating as much information as possible. It should be clear now that in theory they are a strong tool but not so often one is able to solve the extension problem or finding the stable page. Additional structure will provide more ways to reach the stable page or to find the right extension. We start recalling the multiplicative structure of a spectral sequence.

Definition 4.0.4 (Multiplicative Structure). A cohomology spectral sequence $E_2^{p,q} \Rightarrow C^*$ is called *multiplicative* if the following properties hold:

1. all the $(E_r^{s,t}, d_r)$ are bigraded algebras, i.e. $E_r^{s,t} \cdot E_r^{u,v} \subset E_r^{s+u, t+v}$
2. C^* is a filtered algebra, i.e. $F^p C^m \cdot F^q C^n \subset F^{p+q} C^{m+n}$
3. d_r is a derivation and E_r induces the product of E_{r+1} for all $r \geq 2$.
4. C^* induces the product of E_∞ .

The last point is especially important, since it correlates the product on the stable page to the product (which in a lot of computations we don't know) of the target object C^* .

Proposition 4.0.5. *The Atiyah-Hirzebruch Spectral Sequence (from now on AHSS) is multiplicative and on the second page the algebra structure is given by the usual cup product in singular cohomology.*

Proof. See [1] prop. 4.2.9 □

Multiplicativity will play a huge role in our upcoming computations, apart from the fact that it will let us find the cohomology ring of certain spaces, but property (3) will be crucial in finding the stable page. The last property we want to address here is the presence of a pairing in the AHSS between the homology and cohomology version. The pairing will translate what we know about cohomology in homology which is somewhat harder to compute via spectral sequence. Even though the first feeling with cohomology is that it is much richer and therefore harder to compute, the additional structure will play a huge role in finding out stable page and limit in the AHSS. On the contrary, since homology doesn't have much structure, trying to compute it via the spectral sequence directly without using the pairing is very hard. If E is a ring spectrum (we will denote the (co)homology induced by it with \mathbb{E}^* and \mathbb{E}_*) and X is a CW-complex, then there is a pairing

$$\langle -, - \rangle: \mathbb{E}^n(X) \otimes \mathbb{E}_m(X) \rightarrow \mathbb{E}_{m-n}(\text{pt.})$$

defined by letting $\langle f, g \rangle$ be represented by

$$S^m \xrightarrow{g} E \wedge X^+ \xrightarrow{1 \wedge f} E \wedge \Sigma^n E \xrightarrow{\Sigma^n \mu} \Sigma^n E$$

Observe that for groups G, G' evaluation of cochains on chains induces a map

$$H^n(X; G) \otimes H_n(X; G') \rightarrow G \otimes G'$$

Composing with a group homomorphism $G \otimes G' \rightarrow G''$ gives a pairing

$$\langle -, - \rangle: H^n(X; G) \otimes H_n(X; G') \rightarrow G''$$

Thus, if E is a ring spectrum, the map $\mu_*: \mathbb{E}^{-s}(\text{pt.}) \otimes \mathbb{E}_t(\text{pt.}) \rightarrow \mathbb{E}_{s+t}(\text{pt.})$ defines a pairing

$$\langle -, - \rangle: H^n(X; \mathbb{E}^{-s}(\text{pt.})) \otimes H_n(X; \mathbb{E}_t(\text{pt.})) \rightarrow \mathbb{E}_{s+t}(\text{pt.}) \tag{3}$$

Proposition 4.0.6. *Let E be a ring spectrum, and let X be a CW-complex such that $X = X^N$ for some natural number N . Assume that $\mathbb{E}_*(pt.)$ is bounded below. Consider the AHSS*

$$E_{n,t}^2 \cong H_n(X; \mathbb{E}_t(pt.)) \implies \mathbb{E}_{n+t}(X)$$

and

$$E_2^{n,t} = H^n(X; \mathbb{E}^t(pt.)) \implies \mathbb{E}^{n+t}(X)$$

then there is a natural pairing

$$\langle -, - \rangle: E_r^{n,-s} \otimes E_{n,t}^r \rightarrow \mathbb{E}_{s+t}(pt.)$$

such that

1. the pairing on $E_2 \otimes E^2$ is the pairing of (3)
2. $\langle d_r(x), y \rangle = \langle x, d^r(y) \rangle$ for all $x \in E_r^{n,-s}$ and $y \in E_{n+r,t-r+1}^r$
3. the pairing on $\mathbb{E}^*(X) \otimes \mathbb{E}_*(X)$ induces the pairing on $E_\infty \otimes E^\infty$

Proof. See [1] Proposition 4.2.10, page 129. □

Before concluding this section, we will define here the edge homomorphism and we will identify them in the case of the AHSS. Edge homomorphisms are maps which arise if there is a *transition* from a quadrant with only zeroes and a possible non trivial quadrant of a spectral sequence. So a first quadrant spectral sequence will have two edge homomorphisms, in general AHSS has only one edge homomorphism. So let us consider a spectral sequence whose second and third quadrant are zero:

	\vdots	\vdots	\vdots	\vdots	\vdots
	$E_2^{0,3}$	$E_2^{1,3}$	$E_2^{2,3}$	$E_2^{3,3}$	$E_2^{4,3}$
	$E_2^{0,2}$	$E_2^{1,2}$	$E_2^{2,2}$	$E_2^{3,2}$	$E_2^{4,2}$
	$E_2^{0,1}$	$E_2^{1,1}$	$E_2^{2,1}$	$E_2^{3,1}$	$E_2^{4,1}$
	$E_2^{0,0}$	$E_2^{1,0}$	$E_2^{2,0}$	$E_2^{3,0}$	$E_2^{4,0}$
	$E_2^{0,-1}$	$E_2^{1,-1}$	$E_2^{2,-1}$	$E_2^{3,-1}$	$E_2^{4,-1}$
	\vdots	\vdots	\vdots	\vdots	\vdots

Since there are no non-trivial differentials hitting the 0th column, the stable objects there are simply subobjects of $E_2^{0,p}$. Using the fact that $E_\infty^{0,n} \cong \mathbb{E}^n(X) / F^1 \mathbb{E}^n(X)$ we can fit all these information in the maps

$$\mathbb{E}^n(X) \rightarrow E_\infty^{0,n} \hookrightarrow E_2^{0,n}$$

It is clear that if the edge homomorphism is surjective, the last arrow is the identity and therefore $E_\infty^{0,n} = E_2^{0,n}$. This means that all the differentials starting from element in such column have to vanish.

In conclusion, one can prove that the edge homomorphism in the case of AHSS is simply the map induced by the inclusion $pt. \hookrightarrow X$, and therefore it is always surjective!

Relative Spectral Sequences

Recall that, for any (co)homology theory h_* , the equality $h_*(X, A) \cong \tilde{h}_*(X/A)$ holds. Therefore in order to compute relative (co)homology groups for a pair of CW complexes it is enough to be able to compute relative (co)homology groups. So we give here the following theorem

Theorem 4.0.7. *Let $F \hookrightarrow E \xrightarrow{p} B$ be a fibration with B a CW-complex. Let $A \subset B$ be a subcomplex. Let $D = p^{-1}(A)$. Let G be an unreduced (co)homology theory*

- *There is a homology spectral sequence with*

$$E_{p,q}^2 = H_p(B, A; G_q(F)) \Rightarrow G_{p+q}(E, D)$$

- *If B is finite dimensional or if there exists an N so that $G^q(F) = 0$ for all $q < N$, there is a cohomology spectral sequence with*

$$E_2^{p,q} = H^p(B, A; G^q(F)) \Rightarrow G^{p+q}(E, D)$$

Proof. See [3] page 266 Theorem 9.33 or [4] page 351 Remark 2. □

Taking $A = *$ and $p = \text{Id}$ we obtain our statement for reduced (co)homology theories.

5 Generalized Orientation for Manifold, Bundles and (B, f) -Structures

Let us consider a (B, f) -structure \mathfrak{B} . Let (M^n, e, g) be a smooth manifold with a \mathfrak{B} -structure and let E be a ring spectrum. Recall that we have the following commutative diagram:

$$\begin{array}{ccc} & & B_m \\ & \nearrow g_m & \downarrow f_m \\ M^n & \xrightarrow{\nu_e} & BO(m) \end{array}$$

Consider now the induced map $g^*: \mathbb{E}^*(B_m) \rightarrow \mathbb{E}^*(M^n)$. If $\xi \in \mathbb{E}^k(B_m)$, then

$$g^*(\xi) \in \mathbb{E}^k(M^n)$$

is called the ξ -characteristic class of (M^n, e, g) .

We generalize now the definition of an oriented manifold:

Definition 5.0.1. A manifold M^n is called \mathbb{E} -oriented if there is a class $\iota \in \mathbb{E}_n(M^n, \partial M^n)$, called an \mathbb{E} -fundamental class of M^n , which maps to a generator of $\tilde{\mathbb{E}}_n(S^n)$ under the canonical map

$$\mathbb{E}_n(M^n, \partial M^n) \xrightarrow{j_*} \mathbb{E}_n(M^n, M^n \setminus \{m\}) \cong \mathbb{E}_n(D(m), S(m)) \cong \tilde{\mathbb{E}}_n(S^n)$$

for each $m \in \text{Int}M^n$.

Remark 5.0.2. It's important to clarify what we mean by generator here. A generator for $\tilde{\mathbb{E}}_n(S^n)$ is an element $x \in \tilde{\mathbb{E}}_n(S^n)$ such that there exists a unit $\xi \in \pi_0 E$ such that $x = \xi \Sigma^n \iota$, where we denoted by ι the class represented by ι , the unit morphism $\mathbb{S} \rightarrow E$.

Remark 5.0.3. Note that if we take the Eilenberg-MacLane spectrum $K(\mathbb{Z})$, a $K(\mathbb{Z})$ -oriented manifold is an oriented manifold in the usual sense.

Definition 5.0.4. If (M^n, e, g) is E -oriented, we define the ξ -characteristic number of (M^n, e, g) by

$$\xi(M^n, e, g) := \langle g^*(\xi), \iota \rangle \in \mathbb{E}_{n-k}(\text{pt.})$$

where we used the pairing between $\mathbb{E}^*(M^n)$ and $\mathbb{E}_*(M^n)$.

Let $B = \operatorname{colim}_k B_k$ be the colimit along the maps $g_k: B_k \rightarrow B_{k+1}$. If all manifolds with \mathfrak{B} -structure are E -oriented then we have a *characteristic number map*,

$$\begin{aligned} e: \mathbb{E}^k(B) \otimes \Omega_n^{\mathfrak{B}} &\rightarrow E_{n-k}(\text{pt.}) \\ (\xi \otimes [M^n, e, g]) &\mapsto p_{n+m}^*(\xi)(M^n, e, g) \end{aligned}$$

where $p_{n+m}: B_{n+m} \rightarrow B$ is the canonical map given by the colimit construction of B . When $[N^n, e, g] = \partial[W^{n+1}, \epsilon, G]$ the E -fundamental class $\iota_W \in \mathbb{E}_{n+1}(W^{n+1}, N^n)$ satisfies $\partial(\iota_W) = \iota_N$, where ∂ is the connecting homomorphism in homology. Consider the following long exact sequence where $j: M^n \rightarrow W^{n+1}$ is the inclusion map.

$$\xi(N^n, e, g) = \langle g^*(\xi), \iota_N \rangle = \langle j^*G^*(\xi), \partial(\iota_W) \rangle = \langle G^*(\xi), j_*\partial(\iota_W) \rangle = 0$$

Therefore the ξ -characteristic number doesn't depend on the representative chosen to represent the bordism class of $[M^n, e, g]$.

We now define E -oriented vector bundles and E -oriented (B, f) -structure.

Definition 5.0.5. Let us consider a (B, f) -structure \mathfrak{B} and let E be a ring spectrum.

1. Let $\xi: X \rightarrow Y$ be a k -dimensional vector bundle. Consider Y to be the subset of X given by the zero section. Then ξ is called \mathbb{E} -oriented if there is an element $U \in \mathbb{E}^k(X, X \setminus Y) \cong \tilde{\mathbb{E}}^k(M(\xi))$, called the Thom class of ξ , such that the restriction of U to

$$\mathbb{E}^k(\xi^{-1}(y), \xi^{-1}(y) \setminus \{y\}) \cong \tilde{\mathbb{E}}^k(S^k)$$

is a generator of $\tilde{\mathbb{E}}^k(S^k)$ over π_*E for all $y \in Y$

2. A vector bundle $\xi: X \rightarrow Y$ has a \mathfrak{B} -structure if there is a map $g: Y \rightarrow B_k$ such that ξ is the pullback via $f_k \circ g: Y \rightarrow B_k \rightarrow BO(k)$ of the canonical bundle over $BO(k)$.
3. \mathfrak{B} is called \mathbb{E} -oriented if the pullback $\pi_m^{\mathfrak{B}}$ of the canonical bundle over $BO(m)$ along f_m is \mathbb{E} -oriented for every m .
4. An \mathbb{E} -orientation for a (B, f) -structure \mathfrak{B} is a collection of Thom classes of $\pi_m^{\mathfrak{B}}$ in $\tilde{\mathbb{E}}^m(M(\pi_m^{\mathfrak{B}}))$ for every m .

Proposition 5.0.6. We collect here some properties of an \mathbb{E} -orientation of a (B, f) -structure.

1. The pullback of an \mathbb{E} -oriented vector bundle is \mathbb{E} -oriented
2. If \mathfrak{B} is \mathbb{E} -oriented, then the normal bundle of every manifold with \mathfrak{B} -structure is \mathbb{E} -oriented.
3. For any multiplicative (B, f) -structure \mathfrak{B} , the identity map of $M\mathfrak{B}$ defines a $M\mathfrak{B}$ -orientation of \mathfrak{B} .
4. If $\alpha: \mathfrak{B} \rightarrow \mathfrak{B}'$ is a map of multiplicative (B, f) -structures then $M\alpha: M\mathfrak{B} \rightarrow M\mathfrak{B}'$ defines a $M\mathfrak{B}'$ -orientation of \mathfrak{B} .
5. BU is MSO -oriented, BSp is MU -oriented and EO is $M\mathfrak{B}$ -oriented for any (B, f) -structure \mathfrak{B} .

Proof. 1. Let $\xi: X \rightarrow Y$ be an \mathbb{E} -oriented n -bundle. Let $f: Z \rightarrow Y$ any continuous map. Let $u_\xi \in \tilde{\mathbb{E}}^n(M(\xi))$ be a Thom class for ξ . We have the following commutative diagram by definition of pullback bundle

$$\begin{array}{ccc} X' & \xrightarrow{\hat{f}} & X \\ f^*\xi \downarrow & & \downarrow \xi \\ Z & \xrightarrow{f} & Y \end{array}$$

where \hat{f} is a fibre-wise isomorphism. Now we apply the functor $M: \mathbf{VectBun} \rightarrow \mathbf{Top}$ which gives us the Thom space of a vector bundle and then the cohomology functor $\tilde{\mathbb{E}}^n$

$$\begin{array}{ccc}
\tilde{\mathbb{E}}^n(M(f^*\xi)) & \xleftarrow{\tilde{\mathbb{E}}^n(M\hat{f})} & \tilde{\mathbb{E}}^n(M(\xi)) \\
\downarrow \tilde{\mathbb{E}}^n(M(\text{incl.})) & & \downarrow \tilde{\mathbb{E}}^n(M(\text{incl.})) \\
\tilde{\mathbb{E}}^n(S^n) & \xleftarrow{\tilde{\mathbb{E}}^n(M(\hat{f}\circ\text{incl.}))} & \tilde{\mathbb{E}}^n(S^n)
\end{array}$$

where the lower horizontal arrow is an isomorphism by construction and therefore the class $\tilde{\mathbb{E}}^n(M\hat{f})u_\xi$ is a Thom class for $f^*\xi$ by commutativity of the diagram.

2. By def. of a n -manifold with (B, f) -structure, the normal bundle is the pullback of the bundle $\pi_n^{\mathfrak{B}}$. This one is oriented by assumption and therefore by the preceding point we are done.
3. Recall that we added the assumption of being a multiplicative (B, f) -structure since we want to work with a multiplicative spectrum. The identity defines a family of maps $\text{Id}_n: M\mathfrak{B}_n \rightarrow M\mathfrak{B}_n$ which in turn represent elements $u_n \in \widetilde{M\mathfrak{B}}^n(M\mathfrak{B}_n)$. In fact,

$$\begin{aligned}
\text{Id}_n &: M\mathfrak{B}_n \rightarrow M\mathfrak{B}_n \\
\text{Id}_n &: \Sigma^\infty M\mathfrak{B}_n \rightarrow \Sigma^\infty M\mathfrak{B}_n \\
\text{Id}_n &: \Sigma^\infty M\mathfrak{B}_n \rightarrow \Sigma^\infty M\mathfrak{B}_n \subseteq \Sigma^n M\mathfrak{B}
\end{aligned}$$

So we need to prove that for every $n \in \mathbb{N}$, u_n is the Thom class of $\pi_n^{\mathfrak{B}}$. Consider the map of ring spectra induced by inclusion of a fibre, we showed on Tuesday when speaking about (B, f) -structures that it represents the unit $\iota: \mathbb{S} \rightarrow M\mathfrak{B}$, and the restriction via the inclusion $\iota_n: S^n \rightarrow M\mathfrak{B}_n$ is exactly $\Sigma^n \iota$, therefore it is a generator by definition (we are restricting the map induced by the identity in degree n which is the identity).

4. We proceed as in the previous point using the properties of a map of multiplicative (B, f) -structures.
5. There are maps of multiplicative (B, f) structures

$$EO \rightarrow BSp \rightarrow BU \rightarrow BSO \rightarrow BO$$

induced by the natural inclusion of subgroups $\{1\} \subset Sp_n \subset U_n$ (after choosing as a model for $B\{e\}$ the space EO), $U_m \subset SO(2m)$ and $SO(k) \subset O(k)$. Recall then that we require that there exists a map of (B, f) -structure $EO \rightarrow \mathfrak{B}$ for any \mathfrak{B} , apply then the previous point. \square

We show in the next lemma that a manifold is \mathbb{E} -orientable if and only if its normal bundle is \mathbb{E} -oriented. The proof uses the following cap product construction. Let E be a ring spectrum, and let A, B be two subcomplexes of a CW -complex X . Define the cap product

$$\frown: \mathbb{E}^s(X, A) \otimes \mathbb{E}_{s+t}(X, A \cup B) \rightarrow \mathbb{E}_t(X, B)$$

by letting $[x] \frown [u]$ be represented by

$$S^{s+t} \xrightarrow{u} E \wedge X / A \cup B \xrightarrow{1 \wedge \Delta} E \wedge X / A \wedge X / B \xrightarrow{1 \wedge x \wedge 1} E \wedge \Sigma^s E \wedge X / B \xrightarrow{\mu \wedge 1} \Sigma^s E \wedge X / B$$

where $\Delta: X / A \cup B \rightarrow X / A \wedge X / B$ is the diagonal map, $x: X \cup CA \rightarrow \Sigma^s E$ represents $[x] \in \mathbb{E}^s(X, A)$ and $u: S^{s+t} \rightarrow E \wedge (X \cup C(A \cup B))$ represents $[u] \in \mathbb{E}_{s+t}(X, A \cup B)$

Proposition 5.0.7. *The cap product is natural, i.e. for $f: (X; A, B) \rightarrow (X'; A', B')$ a continuous map, the relation*

$$f_*(f^*x' \frown u) = x' \frown f_*u$$

holds

Proof. The element $f_*(f^*x' \frown u)$ is represented by the composition

$$S^{s+t} \xrightarrow{u} E \wedge X / A \cup B \xrightarrow{1 \wedge \Delta} E \wedge X / A \wedge X / B \xrightarrow{1 \wedge f \wedge 1} E \wedge X' / A' \wedge X / B \xrightarrow{1 \wedge x \wedge 1} \\ \xrightarrow{1 \wedge x \wedge 1} E \wedge \Sigma^s E \wedge X / B \xrightarrow{\mu \wedge 1} \Sigma^s E \wedge X / B \xrightarrow{1 \wedge f} \Sigma^s E \wedge X' / B'$$

now notice that we can *pull-back* the rightmost f since is precomposed with identities to obtain

$$S^{s+t} \xrightarrow{u} E \wedge X / A \cup B \xrightarrow{1 \wedge \Delta} E \wedge X / A \wedge X / B \xrightarrow{1 \wedge f \wedge f} E \wedge X' / A' \wedge X' / B' \xrightarrow{1 \wedge x \wedge 1} \\ \xrightarrow{1 \wedge x \wedge 1} E \wedge \Sigma^s E \wedge X' / B' \xrightarrow{\mu \wedge 1} \Sigma^s E \wedge X' / B'$$

now using the easy equality $(f \wedge f) \circ \Delta = \Delta \circ f$ we obtain the representative for $x' \frown f_*u$, namely

$$S^{s+t} \xrightarrow{u} E \wedge X / A \cup B \xrightarrow{1 \wedge f} X' / A' \cup B' \xrightarrow{1 \wedge \Delta} E \wedge X' / A' \wedge X' / B' \xrightarrow{1 \wedge x \wedge 1} \\ \xrightarrow{1 \wedge x \wedge 1} E \wedge \Sigma^s E \wedge X' / B' \xrightarrow{\mu \wedge 1} \Sigma^s E \wedge X' / B'$$

□

Proposition 5.0.8. *Let E be a ring spectrum. Let $e: M^n \hookrightarrow \mathbb{R}^{n+k}$ be an embedding of a compact smooth manifold M^n without boundary into an $(n+k)$ -dimensional subspace of \mathbb{R}^∞ . Let $N(M^n)$ be a closed tubular neighbourhood of M^n in \mathbb{R}^{n+k} with $\nu: N(M^n) \rightarrow M^n$ the normal bundle. Then the following are equivalent:*

- the vector bundle ν is \mathbb{E} -oriented.
- M^n is \mathbb{E} -oriented.

Proof. (1) **implies** (2). Let $c: S^{n+k} \rightarrow N(M^n) / \partial N(M^n)$ denote the (pointed) map Pontrjagin-Thom collapse map. We recall how it was defined here:

$$c: S^{n+k} \xrightarrow{\cong} (\mathbb{R}^{n+k})^* \xrightarrow{\dagger} \text{Th}(\nu) \cong N(M^n) / \partial N(M^n)$$

where the dagger map is simply (with a little abuse of notation)

$$x \mapsto \begin{cases} x & \text{if } x \in N(M^n) \\ \text{basepoint} & \text{if } x \notin N(M^n) \end{cases}$$

We make the identifications

$$\tilde{\mathbb{E}}_* \left(T(M^n) / \partial T(M^n) \right) \cong \mathbb{E}_*(T(M^n), \partial T(M^n)) \cong \mathbb{E}_*(T(M^n), T(M^n) \setminus M^n)$$

Let now $\iota_{k+n} \in \tilde{E}_{k+n}(S^{n+k})$ be an π_*E -generator of $\tilde{E}_{k+n}(S^{n+k})$, and let u_ν be an \mathbb{E} -orientation of ν . Define

$$\iota_M := \nu_*(u_\nu \frown c_*(\iota_{k+n}))$$

We show now that ι_M is an \mathbb{E} -fundamental class of M^n . If $m \in M^n$, let U be a closed neighbourhood of m homeomorphic to a closed disk D^n such that $\nu^{-1}(U) \cong U \times \mathbb{R}^k$. We identify $\nu^{-1}(U)$ with $U \times \mathbb{R}^k$ to ease the notation. We show now that such ι_M is an \mathbb{E} -fundamental class of M^n . We will do it by proving that for each $m \in M^n$, ι_M maps to a π_*E -generator of $\mathbb{E}_n(M^n, M^n \setminus \{m\})$.

Consider now the following diagram

$$\begin{array}{ccccc}
& & \tilde{\mathbb{E}}_{n+k}(S^{n+k}) & & \\
& \swarrow c_* & & \searrow c'_* & \\
\tilde{\mathbb{E}}_{n+k}(D(\nu)/S(\nu)) & & & & \tilde{\mathbb{E}}_{n+k}((U \times D^k)/\partial(U \times D^k)) \\
\cong \uparrow & & & & \cong \uparrow \\
\mathbb{E}_{n+k}(D(\nu), S(\nu)) & \xrightarrow{\text{incl.}} & \mathbb{E}_{n+k}(D(\nu), S(\nu) \cup D') & \xleftarrow{\text{incl.}} & \mathbb{E}_{n+k}((U, \partial U) \times (D^k, S^{k-1})) \\
\downarrow u_\nu \frown - & & \downarrow u_\nu \frown - & & \cong \downarrow u'_\nu \frown - \\
\mathbb{E}_n(D(\nu)) & \xrightarrow{\text{incl.}} & \mathbb{E}_n(D(\nu), D') & \xleftarrow{\text{incl.}} & \mathbb{E}_n((U, \partial U) \times D^k) \\
\cong \downarrow \nu_* & & \cong \downarrow \nu_* & & \cong \downarrow \nu_* \\
\mathbb{E}_n(M^n) & \xrightarrow{\text{incl.}} & \mathbb{E}_n(M^n, M^n \setminus \text{Int}U) & \xleftarrow{\cong \text{incl.}} & \mathbb{E}_n(U, \partial U) \\
& \searrow \text{incl.} & \cong \downarrow \text{incl.} & & \\
& & \mathbb{E}_n(M^n, M^n \setminus \{m\}) & &
\end{array}$$

In this diagram

$$c': S^{n+k} \rightarrow U \times D^k / \partial(U \times D^k)$$

is the map which collapses the exterior of $U \times D^k$ to a point,

$$D' = \nu|_{D(\nu)}^{-1}(M^n \setminus \text{Int}U)$$

and $u'_\nu \in \mathbb{E}^k(U \times (D^k, S^{k-1}))$ is the canonical generator of $\mathbb{E}^*(U \times (D^k, S^{k-1})) \cong \mathbb{E}^*(D^k, S^{k-1})$ over π_*E .

Claim 5.0.9. *The diagram is commutative*

Proof. The proof is pretty tedious, the first commutative square to check (more appropriately, the pentagon on the top) commutes since at the end we land in $\mathbb{E}_{n+k}(D(\nu), S(\nu) \cup D') \cong \tilde{\mathbb{E}}_{n+k}(D(\nu)/S(\nu) \cup D')$ and therefore we can just forget what happens outside of $S(\nu) \cup D'$, in particular, the right path shows what's really relevant. The two squares which lies just below this pentagon commutes by naturality of the cap product. The remaining squares commutes by the naturality of the l.e.s. of the respective pairs. \square

Claim 5.0.10. *the map $c': S^{n+k} \rightarrow U \times D^k / \partial(U \times D^k)$ is a homeomorphism*

Proof. Recall that $U \cong D^n$, the rest is trivial. \square

Claim 5.0.11. *The map $u'_\nu \frown -: \mathbb{E}_{n+k}((U, \partial U) \times (D^k, S^{k-1})) \rightarrow \mathbb{E}_n((U, \partial U) \times D^k)$ is an isomorphism*

Proof. For a trivial bundle, capping with the Thom class coincide with the suspension isomorphism map. See [2] page 249 Prop. 10.2.5 \square

Now since the left-external path represents the restriction of i_n via the inclusion and the right path is a composition of isomorphisms, the first implication is proved.

(2) **implies** (1) Since M^n is compact we can write

$$M^n = N_{m_1} \cup \dots \cup N_{m_s}$$

where each N_{m_i} is a trivialized neighbourhood of M diffeomorphic to an open disk D^n .

Claim 5.0.12. *We have a natural map*

$$\text{hom}_{\mathbb{E}_*}(\mathbb{E}_n(N_{m_i}, M_{m_i} \setminus \{m_i\}), \mathbb{E}_0(pt.)) \xrightarrow{\cong} \mathbb{E}^n(N_{m_i}, N_{m_i} \setminus \{m_i\})$$

Proof. Recall that we can identify $\mathbb{E}_n(N_{m_i}, M_{m_i} \setminus \{m_i\}) \cong \widetilde{\mathbb{E}}_n(N_{m_i}/M_{m_i} \setminus \{m_i\})$ and that by construction $N_{m_i}/M_{m_i} \setminus \{m_i\} \cong S^n$. Therefore for any $f \in \widetilde{\mathbb{E}}_n(N_{m_i}/M_{m_i} \setminus \{m_i\})$ we have:

$$f: \pi_n(\Sigma^\infty S^n \wedge E) \rightarrow \pi_0(E) \quad (4)$$

$$f: [\Sigma^n \mathbb{S}, \Sigma^\infty S^n \wedge E] \rightarrow [\mathbb{S}, E] \quad (5)$$

$$f: [\mathbb{S}, E] \rightarrow [\mathbb{S}, E] \quad (6)$$

Where in passage (5) we used the fact that $\Sigma^{-n} \Sigma^\infty S^n \simeq \mathbb{S}$. Since f is assumed to commute with \mathbb{E}_* , we have that its uniquely determined by its value at the unit $\iota: \mathbb{S} \rightarrow E$. Therefore the hom set in question is isomorphic to $[\mathbb{S}, E] = \mathbb{E}^0(\text{pt.})$ which, after following backwards the usual identifications, turns out to be (naturally) isomorphic with $\mathbb{E}^n(N_{m_i}, N_{m_i} \setminus \{m_i\})$. Naturality comes from the fact that the suspension isomorphism is natural and when passing from the hom set of homology to cohomology we are basically using the identity. \square

Now we can consider the following chain of identifications:

$$\begin{aligned} & \mathbb{E}^n(N_{m_i}, N_{m_i} \setminus \{m_i\}) \\ & \quad \downarrow \cong \\ & \widetilde{\mathbb{E}}^{n+k}(\Sigma^k(N_{m_i}/N_{m_i} \setminus \{m_i\})) \\ & \quad \downarrow \cong \\ & \mathbb{E}^{n+k}(N_{m_i} \times \mathbb{R}^k, N_{m_i} \times (\mathbb{R}^k \setminus \{0\})) \\ & \quad \downarrow \cong \\ & \mathbb{E}^{n+k}(\nu^{-1}(N_{m_i}), \nu^{-1}(N_{m_i}) \setminus N_{m_i}) \end{aligned}$$

where q^* is induced by $q: N_{m_i} \times \mathbb{R}^k / N_{m_i} \times (\mathbb{R}^k \setminus \{0\}) \rightarrow S^k \wedge N_{m_i} / N_{m_i} \setminus \{m_i\}$ the quotient map, together with the usual identification of the cohomology of a pair with the reduced one of the quotient. The map φ_i^* is induced by a local trivialization of the normal bundle of M . Since all these identifications are natural, by starting with the image of $i_* \iota_N$ (thanks to Claim 5.0.12) in the top group, we get a class in the lower group which is natural w.r.t. restrictions, since $i_* \iota_N$ is natural w.r.t. the restrictions (coming from a global class). Let us denote with $V_i \in \mathbb{E}^{n+k}(\nu^{-1}(N_{m_i}), \nu^{-1}(N_{m_i}) \setminus N_{m_i})$ the class obtained via this process. Using induction together with a Mayer-Vietoris sequence argument we can glue those V_i 's together (the class are natural) and obtain a global class $V \in \mathbb{E}^{n+k}(\nu^{-1}(M), \nu^{-1}(M) \setminus M)$ which clearly restrict to a generator on each fiber, hence the vector bundle ν is E -oriented. \square

6 Thom isomorphism in generalized cohomology

It's easy to see that the construction of a generalized (co)-homology theory carries over to the setting of spectra and not just pointed CW -complexes (for example see [6]). If we work with the Eilenberg-MacLane spectrum HG , for some abelian group G , we get the following equivalent definition for the homology $H_*(E; G)$ of a spectrum:

$$H_n(E; G) := \text{colim}_k H_{n+k}(E_k; G)$$

The Hurewicz homomorphism in generalized homology is defined just as the one in ordinary homology by sending an element of $\pi_* X$ to its value on the fundamental class. We have the following results:

Theorem 6.0.1. (i) *Let E be a spectrum with $\pi_i(E) = 0$ for $i < n$. Then $H_i(E) = 0$ for $i < n$, and the Hurewicz homomorphism $h: \pi_k(E) \rightarrow H_k(E, \mathbb{Z})$ is an isomorphism for $k = n$ and an epimorphism for $k = n + 1$.*

(ii) *For every spectrum E and every abelian group A , there are exact sequences*

$$0 \rightarrow \text{Ext}(H_{n-1}(E; \mathbb{Z}), A) \rightarrow H^n(E; A) \rightarrow \text{hom}_{\mathbb{Z}}(H_n(E), A) \rightarrow 0$$

and

$$0 \rightarrow H_n(E) \otimes A \rightarrow H_n(E; A) \rightarrow \text{Tor}(H_{n-1}(E); A) \rightarrow 0$$

Proof. See [6] Corollary 4.7 and Theorem 4.8 page 82. \square

We are ready to prove the main result of this class, i.e. the Thom Isomorphism Theorem for a \mathbb{E} -oriented vector bundle.

Proposition 6.0.2. *Let E be a connective ring spectrum, and let $\xi: X \rightarrow Y$ be an \mathbb{E} -oriented n -dimensional vector bundle with Thom class $u_\xi \in \mathbb{E}^n(X, X \setminus Y)$.*

1. *The map*

$$\begin{aligned} \Phi_*: \tilde{\mathbb{E}}_{t+n}(M\xi) &\cong \mathbb{E}_{t+n}(X, X \setminus Y) \rightarrow \mathbb{E}_t(Y) \\ z &\mapsto \xi_*(u_\xi \frown z) \end{aligned}$$

is an isomorphism

2. *If Y is a finite dimensional CW complex, then the map*

$$\begin{aligned} \Phi^*: \mathbb{E}^t(Y) &\rightarrow \mathbb{E}^{t+n}(X, X \setminus Y) \cong \tilde{\mathbb{E}}^{t+n}(M\xi) \\ w &\mapsto u_\xi \smile \xi^*(w) \end{aligned}$$

is an isomorphism

Proof. 1. Notice that by definition $\mathbb{E}_0(*) = \pi_0 E = H_0(E; \mathbb{Z})$. Let us consider $H^0(E; \pi_0(E))$. Since our spectrum E is connective by assumption, by Theorem 6.0.1.i we have that $H_{-1}(E; \mathbb{Z}) = 0$ and therefore by UCT we have, for every abelian group A , an isomorphism $H^0(E; A) \cong \text{hom}_{\mathbb{Z}}(\pi_0(E), A)$. Therefore if we set $A = \pi_0(E)$ we have

$$H^0(E; \pi_0(E)) \cong \text{hom}_{\mathbb{Z}}(\pi_0(E), \pi_0(E))$$

Now let us consider the following chain of isomorphisms

$$[E, K(\pi_0(E))] \cong H^0(E; \pi_0(E)) \cong \text{hom}_{\mathbb{Z}}(\pi_0(E); \pi_0(E))$$

where $K(\pi_0(E))$ is the Eilenberg-MacLane spectrum associated to the abelian group $\pi_0(E)$. Define $J: E \rightarrow K(\pi_0(E))$ corresponding to the identity in the latter group.

Claim 6.0.3. ξ *is $K(\pi_0 E)$ -oriented.*

Proof. A map of spectra induces a natural transformation of cohomology theories, so we have the following commutative diagram

$$\begin{array}{ccccc} \pi_0 E & \xleftarrow{\cong} & \mathbb{E}^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) & \xleftarrow{\text{incl.}} & \mathbb{E}^n(X, X \setminus Y) \\ \downarrow \text{Id} & & \downarrow J & & \downarrow J \\ \pi_0 E & \xleftarrow{\cong} & H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \pi_0(E)) & \xleftarrow{\text{incl.}} & H^n(X, X \setminus Y; \pi_0(E)) \end{array}$$

where commutativity in the square on the left is justify by the very definition of J : we choose it such that the diagram commutes. Therefore $J_*(u_\xi)$ is a Thom class for ξ and so the claim. \square

The cap product

$$\frown: \mathbb{E}^s(X, X \setminus Y) \otimes \mathbb{E}_{s+t}(X, X \setminus Y) \rightarrow \mathbb{E}_t(X)$$

induces a pairing of AHSSes ⁴ such that the pairing $E_2^{s,-p} \otimes E_{s+t,q}^2 \rightarrow E_{t,p+q}^2$ is the classical cap product,

$$\frown: H^s(X, X \setminus Y; \mathbb{E}_p) \otimes H_{s+t}(X, X \setminus Y; \mathbb{E}_q) \rightarrow H_t(X; \mathbb{E}_{p+q})$$

⁴Standard reference: http://www.numdam.org/item?id=SHC_1958-1959__11_2_A10_0, for a discussion <http://mathoverflow.net/questions/231131/cap-product-on-leray-serre-spectral-sequences>

Claim 6.0.4. *Capping with the Thom class induces a map of spectral sequences:*

$$\varphi^2: E_{t+n,k}^2 = H_{t+n}(X, X \setminus Y; E_k(*)) \rightarrow E_{t,k}'^2 = H_t(Y; \mathbb{E}_k(*))$$

Proof. The easiest way to prove this is to observe that

$$\Phi_*: \tilde{\mathbb{E}}_{t+n}(M\xi) \cong \mathbb{E}_{t+n}(X, X \setminus Y) \rightarrow \mathbb{E}_t(Y)$$

induces a map between the exact couples generating the relevant AHSSes. In fact for

$$E_{t+n,k}^2 = H_{t+n}(X, X \setminus Y; E_k(*)) \rightarrow \mathbb{E}_{t+n+k}(X, X \setminus Y)$$

the exact couple is

$$\begin{aligned} D_{s,t} &= \tilde{\mathbb{E}}_{s+t}(M(\xi|_{Y^s})) \\ E_{s,t} &= \mathbb{E}_{s+t}(M(\xi|_{Y^s}), M(\xi|_{Y^{s-1}})) \end{aligned}$$

where we used the fact that the $n + s$ -skeleton of the Thom space is $M(\xi|_{Y^s})$. The maps in the exact couple are the respective inclusions. For

$$E_{t,k}'^2 = H_t(Y; \mathbb{E}_k(*)) \rightarrow \mathbb{E}_{t+k}(Y)$$

the exact couple is

$$\begin{aligned} D_{s,t} &= \mathbb{E}_{s+t}(Y^s) \\ E_{s,t} &= \mathbb{E}_{s+t}(Y^s, Y^{s-1}) \end{aligned}$$

and the maps are again the respective inclusions. Now it should be clear to observe that the Thom *isomorphism* induces a map of exact couples, which implies that it induces a map of AHSSes. \square

Now we need to identify what this map looks like on the second page of our AHSSes.

Consider the map of AHSSes $\varphi^r, r > 2$, induced by $\xi_*(U \frown -)$. Then

$$\varphi^2: E_{t+n,k}^2 = H_{t+n}(X, X \setminus Y; \mathbb{E}_k(*)) \rightarrow E_{t,k}^2 = H_t(Y; \mathbb{E}_k(*))$$

is given by $\varphi^2(Z) = \xi_*(J_*(u_\xi) \frown Z)$. Let us have a look at the reduced AHSS for the Thom space $M(\xi)$

$$\begin{array}{cccccc} \vdots & \vdots & \vdots & \vdots & \vdots & \\ \hline 0 & \cdots & 0 & \tilde{H}^n(M(\xi); \pi_3 E) & \tilde{H}^{n+1}(M(\xi); \pi_3 E) & \\ 0 & \cdots & 0 & \tilde{H}^n(M(\xi); \pi_2 E) & \tilde{H}^{n+1}(M(\xi); \pi_2 E) & \\ 0 & \cdots & 0 & \tilde{H}^n(M(\xi); \pi_1 E) & \tilde{H}^{n+1}(M(\xi); \pi_1 E) & \\ 0 & \cdots & 0 & \tilde{H}^n(M(\xi); \pi_0 E) & \tilde{H}^{n+1}(M(\xi); \pi_0 E) & \\ \hline 0 & \cdots & 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

By looking at the n -th diagonal, we have the identification

$$\mathbb{E}^n(X, X \setminus Y) \cong \widetilde{\mathbb{E}}^n(M(\xi); \pi_0 E) \cong \widetilde{H}^n(M(\xi); \pi_0 E) \cong H^n(X, X \setminus Y; \mathbb{E}^0(*))$$

and by naturality it sends u_ξ to $J_*(u_\xi)$:

$$\begin{array}{ccc} \mathbb{E}^n(X, X \setminus Y) & \xleftarrow{\cong} & H^n(X, X \setminus Y; \mathbb{E}^0(*)) \\ \downarrow \text{incl.}^* & & \downarrow \text{incl.}^* \\ \mathbb{E}^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) & \xleftarrow{\cong} & H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{E}^0(*)) \end{array}$$

Thus φ^2 is the classical Thom isomorphism (which is an isomorphism). By the mapping lemma⁵ we know that since φ^2 is an isomorphism, so are the φ^r , for $r > 2$. By the Comparison Theorem, since Φ_* induces an isomorphism on each page of the AHSSes, then Φ_* itself must be an isomorphism.

2. Consider the maps of AHSSes φ_r , for $r \geq 2$, induced by $u_\xi \smile \xi^*(-)$ and the pairing of AHSSes given by cup product. Then

$$\varphi_2: E_2^{t,k} = H^t(Y, E^k) \rightarrow E_2^{t+n,k} = H^{t+n}(X, X \setminus Y; E^k)$$

is given by $\varphi_2(W) = J_*(u_\xi) \smile \xi^*(W)$. Thus φ_2 is the classical Thom Isomorphism and for the same reasons above, Φ^* is an isomorphism. \square

Recall the following definition:

Definition 6.0.5. Let X and E be spectra. The E_* -Hurewicz homomorphism,

$$h_E: \pi_* X \rightarrow \mathbb{E}_*(X)$$

is defined by $h_E[f] = f_*(\iota)$ where f is an homotopy class $[f: \Sigma^k \mathbb{S} \rightarrow X]$ and $\iota \in \widetilde{\mathbb{E}}_k(S^k)$ is the canonical \mathbb{E}_* -generator.

The diagram in the following proposition exhibits a fundamental relationship between the \mathbb{E}^* -characteristic numbers of a manifold M^n and the value of the Hurewicz homomorphism on the Pontrjagin-Thom construction of M^n .

Proposition 6.0.6. Let E be a ring spectrum, and let \mathfrak{B} be a (B, f) -structure which is \mathbb{E}_* -oriented. Then we have a commutative diagram

$$\begin{array}{ccc} \Omega_*^{\mathfrak{B}} & \xrightarrow[\cong]{PT} & \pi_*(M\mathfrak{B}) \\ \downarrow \tilde{e} & & \downarrow h_E \\ & & \mathbb{E}_*(M\mathfrak{B}) \\ & & \downarrow \Phi_* \cong \\ \text{hom}_{\widetilde{\mathbb{E}}_*}(\mathbb{E}^*(B), \mathbb{E}_*) & \xleftarrow[\zeta]{} & \mathbb{E}_*(B) \end{array}$$

where PT is the Pontrjagin-Thom isomorphism and Φ_* is the Thom isomorphism, \tilde{e} is the adjoint map of the characteristic number map e and ζ is the adjoint of the pairing of $\mathbb{E}^*(B)$ and $\mathbb{E}_*(B)$

Proof. Let $[M^n, e, g]$ be a manifold with \mathfrak{B} -structure, where $e: M^n \rightarrow \mathbb{R}^{n+k}$. Let ι_{n+k} be the canonical generator of $\widetilde{\mathbb{E}}_{n+k}(S^{n+k})$. Observe that $\Phi_* c_*(\iota_{n+k})$ is a fundamental class $\iota_M \in \mathbb{E}_n(M^n)$ of M^n , in fact by

⁵ Exercise 5.2.3 and Theorem 5.2.12 page 125 – 126 on Weibel

definition $\Phi_*c_*(\iota_{n+k}) = \nu_*(\text{th} \frown c_*(\iota_{n+k}))$, where ν is the normal bundle of $e: M^n \rightarrow \mathbb{R}^{n+k}$ and th its Thom class. By prop 5.0.8 the latter is (a possible choice of) ι_M . Therefore:

$$\Phi_*h_{EPT}[M^n, e, g] = \Phi_*h_E(Mg \circ c) \quad (7)$$

$$= \Phi_*(Mg)_*c_*(\iota_{n+k}) \quad (8)$$

$$= g_*\Phi_*c_*(\iota_{n+k}) \quad (9)$$

$$= g_*(\iota_M) \quad (10)$$

If $x \in \mathbb{E}^*(B)$ is any class, then we have:

$$(\zeta\Phi_*h_{EPT}[M^n, e, g])(x) = \langle x, \Phi_*h_E(Mg \circ c) \rangle \quad (11)$$

$$= \langle x, g_*(\iota_M) \rangle \quad (12)$$

$$= e(x \otimes [M^n, e, g]) = (\tilde{e}[M^n, e, g])(x) \quad (13)$$

□

there are important examples in which the canonical map

$$\zeta: \mathbb{E}_*(B) \rightarrow \text{hom}_{\mathbb{E}_*}(\mathbb{E}^*(B), \mathbb{E}_*(\text{pt.}))$$

is a monomorphism. For example, this is the case for $H_*(BO; \mathbb{Z}_2)$, $H_*(BU; \mathbb{Z})$, $H_*(BSp; \mathbb{Z})$ and MU_*BU . When it happens the diagram of the preceding proposition simplifies as follows:

Corollary 6.0.7. *In the situation of the preceding proposition, assume that the pairing of $\mathbb{E}^*(B)$ with $\mathbb{E}_*(B)$ induces a monomorphism*

$$\zeta: \mathbb{E}_*(B) \rightarrow \text{hom}_{\mathbb{E}_*}(\mathbb{E}^*(B), \mathbb{E}_*(\text{pt.}))$$

Then the following diagram commutes.

$$\begin{array}{ccc} \Omega_*^{\mathfrak{B}} & \xrightarrow{PT} & \pi_*(M\mathfrak{B}) \\ \downarrow \zeta^{-1}\tilde{e} & & \downarrow h_E \\ \mathbb{E}_*(B) & \xleftarrow[\cong]{\Phi_*} & \mathbb{E}_*(M\mathfrak{B}) \end{array}$$

Proof. By the diagram in Prop 6.0.6, $\text{Im}\tilde{e} \subseteq \text{Im}\zeta$. Therefore when ζ is a monomorphism, $\zeta^{-1}\tilde{e}$ is well-defined. Thus this corollary follows from Proposition 6.0.6. □

It follows from the preceding corollary that when both ζ and the h_E are monomorphism, a bordism class in $\Omega_*^{\mathfrak{B}}$ is detected by its \mathbb{E} -characteristic numbers.

Corollary 6.0.8. *Let E be a ring spectrum and let \mathfrak{B} be a (B, f) structure which is E -oriented. If ζ and the Hurewicz homomorphism h_E are monomorphism then a closed manifold (M^n, e, g) with \mathfrak{B} -structure is the boundary of a compact $(n+1)$ -manifold with \mathfrak{B} -structure if, and only if, all the \mathbb{E} -characteristic numbers of (M^n, e, g) are zero.*

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