EXAMPLE OF TECHNIQUE 2

Given X =(matchings, paths, trees, cycles, etc.), let $P = \text{conv}(X)$. Let $Q = \{x : Ax \leq b\}$. We want to show that $P = Q$. Showing that $P \subseteq Q$ is usually easy. The other way can be tricky. We saw three different techniques to do so. In the second technique,
a) we first show that Q is bounded,
b) and then we show that every vertex of Q is in X
We illustrate this technique with an example here.

Let

$$X = (\text{perfect matchings in some bipartite graph } G = (V, E)),$$

and

$$P = \text{conv}(X).$$

Let

$$\begin{aligned} Q = \{x \in \mathbb{R}^{|E|} | &\sum_{i:(i,j)\in E} x_{ij} = 1 \ \forall j, \\ &\sum_{j:(i,j)\in E} x_{ij} = 1 \ \forall i, \\ &x_{ij} \geq 0 \ \forall (i,j) \in E\}. \end{aligned}$$

a) We first need to show that Q is bounded. This follows because $0 \leq x_{ij} \leq 1$ for all $(i,j) \in E$, which implies that $Q \subseteq [0,1]^{|E|}$.

b) Now, let x^* be a vertex of Q. Either $x^* \in \{0,1\}^{|E|}$ or there exists $(i,j) \in E$ such that $0 < x^*_{ij} < 1$.

Good case: If $x^* \in \{0,1\}^{|E|}$, then since $x^* \in Q$, it satisfies the equations $\sum x_{ij} = 1$, and so the components $x_{ij} = 1$ correspond to the edges of a perfect matching, meaning that $x^* \in \text{conv}(X)$.

Bad case: Otherwise, if $x^* \notin \{0,1\}^{|E|}$, then there exists $(i,j) \in E$ such that $0 < x^*_{ij} < 1$. Let $E^* = \{(i,j) \in E : 0 < x^*_{ij} < 1\}$. First note that no vertex $v \in V$ can be adjacent to exactly one edge, say (v,w), of E^*. Indeed, $\sum_{u:(u,v)\in E\setminus E^*} x_{uv} = z \in \{0,1\}$, and so $\sum_{u:(u,v)\in E} x_{uv} = x_{vw} + \sum_{u:(u,v)\in E\setminus E^*} x_{uv} = x_{vw} + z$ which is equal to 1 since $x^* \in Q$. If $z = 0$, then $x_{vw} = 1$, and so $(v,w) \notin E^*$. If $z = 1$, then $x_{vw} = 0$, and so $(v,w) \notin E^*$. Therefore, no vertex $v \in V$ can be adjacent to exactly one edge of E^*.

This implies that every vertex of V is adjacent to 0 edges of E^* or at least two edges of E^*. This further implies that there exists a cycle in E^*, and since G is bipartite, it must be an even cycle. Call such a cycle C. Color every other edge in the cycle red, and denote this set of red edges by E_R, and color the remaining edges in the cycle blue, and denote this set of blue edges by E_B. Note that $E_R \cup E_B$ yields all the edges of C.

Let $\epsilon = \min_{(i,j)\in C}\{x^*_{ij}, 1 - x^*_{ij}\}$. Create two new vectors, $x^{(1)}$ and $x^{(2)}$, such that

$$x^{(1)}_{ij} = \begin{cases} x^*_{ij} & \text{if } (i,j) \in E\backslash C \\ x^*_{ij} + \epsilon & \text{if } (i,j) \in E_R \\ x^*_{ij} - \epsilon & \text{if } (i,j) \in E_B \end{cases} \quad \text{and } x^{(2)}_{ij} = \begin{cases} x^*_{ij} & \text{if } (i,j) \in E\backslash C \\ x^*_{ij} - \epsilon & \text{if } (i,j) \in E_R \\ x^*_{ij} + \epsilon & \text{if } (i,j) \in E_B \end{cases}.$$

First observe that $x^{(1)}$ and $x^{(2)}$ are both in Q. Indeed, by the definition of ϵ, $0 \le x^{(1)}_{ij} \le 1$ and $0 \le x^{(2)}_{ij} \le 1$. Moreover, for any vertex v which is not on the cycle, $\sum_{u:(u,v)\in E} x^{(i)}_{uv} = \sum_{u:(u,v)\in E} x^*_{uv} = 1$ for $i = 1$ or 2. For a vertex v on the cycle, v is exactly adjacent to one blue edge and one red edge, so $\sum_{u:(u,v)\in E} x^{(i)}_{uv} = \sum_{u:(u,v)\in E} x^*_{uv} + \epsilon - \epsilon = \sum_{u:(u,v)\in E} x^*_{uv} = 1$ for $i = 1$ or 2. Thus $x^{(1)}$ and $x^{(2)}$ are both in Q.

Then, also observe that $x^* = \frac{1}{2}(x^{(1)} + x^{(2)})$. Indeed, for $e \in E\backslash C$, we get $x^*_e = \frac{1}{2}(x^*_e + x^*_e)$. For $e \in E_R$, we get $x^*_e = \frac{1}{2}(x^*_e + \epsilon + x^*_e - \epsilon)$; similarly, for $e \in E_B$, we get $x^*_e = \frac{1}{2}(x^*_e - \epsilon + x^*_e + \epsilon)$. Thus $x^* = \frac{1}{2}(x^{(1)} + x^{(2)})$.

This means that the vertex x^* is a convex combination of two other points in Q, namely $x^{(1)}$ and $x^{(2)}$, which is impossible. One way to see it is impossible is as follows.

Consider any face F that contains x^*. We'll show that any such face also contains $x^{(1)}$ and $x^{(2)}$.

We know there exists some valid inequality for Q, say $a^\top x \le b$, that induces F. Since x^* is on F, we have that $a^\top x^* = b$. We can rewrite this equation as $\sum_{e\in E} a_e x^*_e = b$.

Since $a^\top x \le b$ is a valid inequality for Q, we know that $a^\top x^{(1)} \le b$ and $a^\top x^{(2)} \le b$ since $x^{(1)}$ and $x^{(2)}$ are in Q. Observe that

$$\begin{aligned} a^\top x^{(1)} &= \sum_{e\in E} a_e x^{(1)}_e \\ &= \sum_{e\in E\backslash C} a_e x^{(1)}_e + \sum_{e\in E_R} a_e x^{(1)}_e + \sum_{e\in E_B} a_e x^{(1)}_e \\ &= \sum_{e\in E\backslash C} a_e x^*_e + \sum_{e\in E_R} a_e (x^*_e + \epsilon) + \sum_{e\in E_B} a_e (x^*_e - \epsilon) \\ &= \sum_{e\in E} a_e x^*_e + \sum_{e\in E_R} a_e \epsilon - \sum_{e\in E_B} a_e \epsilon \\ &= b + y \end{aligned}$$

where $y = \sum_{e\in E_R} a_e \epsilon - \sum_{e\in E_B} a_e \epsilon$.

Similarly,

$$\begin{aligned} a^\top x^{(2)} &= \sum_{e\in E} a_e x_e^{(2)} \\ &= \sum_{e\in E\setminus C} a_e x_e^{(2)} + \sum_{e\in E_R} a_e x_e^{(2)} + \sum_{e\in E_B} a_e x_e^{(2)} \\ &= \sum_{e\in E\setminus C} a_e x_e^* + \sum_{e\in E_R} a_e (x_e^* - \epsilon) + \sum_{e\in E_B} a_e (x_e^* + \epsilon) \\ &= \sum_{e\in E} a_e x_e^* - \sum_{e\in E_R} a_e \epsilon + \sum_{e\in E_B} a_e \epsilon \\ &= b - y \end{aligned}$$

Since both $b + y \leq b$ and $b - y \leq b$ (since $a^\top x \leq b$ is a valid inequality for Q), this implies that $y = 0$, and thus that $a^\top x^{(1)} = b$ and $a^\top x^{(2)} = b$, i.e., that both $x^{(1)}$ and $x^{(2)}$ are on F as well.

Recall that this is true for any face F that contains x^*. In particular, since x^* is a vertex (recall that a vertex is a face) of Q, it is true for that face, i.e. this vertex of dimension 0 contains at least three different points: x^*, $x^{(1)}$, $x^{(2)}$. Thus, it cannot have dimension 0 and x^* cannot be a vertex.

Thus, the bad case is inexistent: no vertex of Q contains non-integral components. All vertices of Q follow the good case, and are all in X.