Worksheet 2.6 (Part 1) - Math 455

1. Suppose a drawer contains three red beads, four blue beads, and five green beads. Use a generating function to determine the number of ways to select six beads if one must select at least one red bead, an odd number of blue beads, and an even number of green beads. Assume that beads of the same color are indistinguishable, and that the order of selection is irrelevant.

   I can take one or two or three red beads and I can take 1 or 3 blue beads and 2 or 4 green beads which can be translated to $G(x) = (x + x^2 + x^3)(x + x^3)(x^0 + x^2 + x^4)$. Expanding, we get $G(x) = x^2 + x^3 + 3x^4 + 2x^5 + 4x^6 + 2x^7 + 3x^8 + x^9 + x^10$. So there are 4 ways of choosing six beads in that way.

2. Use a combinatorial argument to count the number of different five-card hands that can be dealt from a triple deck, then the number of five-card hands that can be dealt from a quadruple deck.

   Three decks: If no cards are repeated, then there are $\binom{52}{5}$ hands. If one card is doubled, I choose four cards out of 52 and then decide which of these four is repeated, so $\binom{52}{4}\binom{4}{1}$. If one card is tripled, I choose three cards out of 52 and then decide which of these three cards is tripled, so $\binom{52}{3}\binom{3}{1}$. It two cards are doubled, I choose three cards out of 52 and decide which two are doubled, so $\binom{52}{3}\binom{3}{2}$. If one card is tripled and one card is doubled, I choose two cards out of 52 and then decide which of these two is doubled, so $\binom{52}{2}\binom{2}{1}$. So in total, I have $\binom{52}{5} + \binom{52}{4}\binom{4}{1} + \binom{52}{3}\binom{3}{1} + \binom{52}{3}\binom{3}{2} + \binom{52}{2}\binom{2}{1}$.

   Four decks: All of the possibilities we had with three decks are still present. On top of that, we might also quadruple a card, in which case I choose two cards out of 52 and then decide which of these two is quadrupled, so $\binom{52}{2}\binom{2}{1}$. So in total, I get $\binom{52}{5} + \binom{52}{4}\binom{4}{1} + \binom{52}{3}\binom{3}{1} + \binom{52}{3}\binom{3}{2} + \binom{52}{2}\binom{2}{1} + \binom{52}{2}\binom{2}{1}$.

3. Use a generating function to determine the number of ways to select a hand of m cards from a triple deck, if there are n distinct cards in a single deck. Verify that your expression produces the correct answers when $n = 52$ and $m = 5$.

   For each card, I can take it zero, one, two or three times. So $G(x) = (x^0 + x^1 + x^2 + x^3)^n$. The coefficient in front of x^m is the number of ways to select a card. We proceed as we did in class for the case of the double deck.

$$
\begin{aligned}
(1 + x + x^2 + x^3)^n &= \sum_{k\geq 0} \binom{n}{k} (x + x^2 + x^3)^k \\
&= \sum_{k\geq 0} \binom{n}{k} x^k (1 + x + x^2)^k \\
&= \sum_{k\geq 0} \binom{n}{k} x^k \sum_{l\geq 0}\sum_{j\geq 0} \binom{k}{l-j}\binom{l-j}{j} x^l \\
&= \sum_{k\geq 0}\sum_{l\geq 0}\sum_{j\geq 0} \binom{n}{k}\binom{k}{l-j}\binom{l-j}{j} x^{k+l} \\
&= \sum_{m\geq 0}\sum_{l\geq 0}\sum_{j\geq 0} \binom{n}{m-l}\binom{m-l}{l-j}\binom{l-j}{j} x^m
\end{aligned}
$$

This somehow looks different from what we found in the previous question for when $n = 52$ and $m = 5$. Note that, had I wanted to be more systematic there, I would have said the following. Let's say that I have a hand of m cards where I have s single cards, d doubled cards and t tripled cards. Then $s+2d+3t = m$ and the number of different cards in your hand is $s+d+t$. So I choose $s+d+t$ cards out of 52, and out of these $s+d+t$ cards, I choose d to be doubled, and out of the $s+t$ remaining cards, I choose t of them to be tripled. So I have $\binom{52}{s+d+t}\binom{s+d+t}{d}\binom{s+t}{t}$ or equivalently $\binom{52}{m-d-2t}\binom{m-d-2t}{d}\binom{m-2d-2t}{t}$.

Let $l = d + 2t$ and $j = t$ in the equation above, and we obtain

$$G(x) = \sum_{m\geq 0}\sum_{d\geq 0}\sum_{t\geq 0}\binom{n}{m-d-2t}\binom{m-d-2t}{d+t}\binom{d+t}{t}.$$

Expanding $\binom{m-d-2t}{d+t}\binom{d+t}{t}$ and $\binom{m-d-2t}{d}\binom{m-2d-2t}{t}$, I see that they are equal.

4. Suppose that an unlimited number of jelly beans is available in each of five different colors: red, green, yellow, white, and black.
   (a) How many ways are there to select twenty jelly beans?
   There are $\binom{5+20-1}{20} = \binom{24}{20}$.
   (b) How many ways are there to select twenty jelly beans if we must select at least two jelly beans of each color?
   Before selecting jelly beans freely, I'll take two of each color. So I really can now only select ten jelly beans in whatever way I want. So $\binom{5+10-1}{10} = \binom{14}{10}$.

5. A catering company brings fifty identical hamburgers to a party with twenty guests.
   (a) How many ways can the hamburgers be divided among the guests, if none is left over?
   There are $\binom{20+50-1}{50} = \binom{69}{50}$.
   (b) How many ways can the hamburgers be divided among the guests, if every guest receives at least one hamburger, and none is left over?
   Before dividing the burgers freely, I first give a burger to every person, so that I am left with 30 burgers to distribute freely between 20 people. So $\binom{20+30-1}{30} = \binom{49}{30}$.
   (c) Repeat these problems if there may be burgers left over.
   There are two ways of thinking of this. Either I can say I am distributing i for any i between 0 and 50, so I get $\sum_{i=0}^{5} 0\binom{20+i-1}{i} = \binom{20+50}{50}$ by problem 5 in 2.2. Or I can think that I am really distributing the burgers among 21 people (where one person gets the any leftovers there might be), and so I get $\binom{21+50-1}{50}$. Similarly for (b), I get $\binom{21+30-1}{30}$.