Worksheet 2.4 - Math 455

1. Show that at any party with at least two people—can it really be a party with less than two people?—there must exist at least two people in the group who know the same number of other guests at the party. Assume that each pair of people at the party are either mutual friends or mutual strangers.

   You can represent that with a graph where the people are vertices, and you put an edge between two people if they are mutual friends. Then the number of other guests each person knows at the party is the degree of the vertex associated to this person. Look at a connected component of your graph on k vertices where $k \geq 2$ (If this doesn't exist, then each vertex has degree 0, and everyone at the party knows zero other guests). The degree of each vertex in that connected component is at least 1 and at most $k-1$. By the pigeonhole principle, two vertices in that connected component must have the same degree, and the two people corresponding to these two vertices must know the same number of people.

2. Wanting to get better at solving math problems, you decide to solve at least one math problem a day and at most 11 math problems per week for one year. Show that there must be a period of consecutive days in the year during which you will solve exactly 20 problems.

   Let a_i be the total number of problems you have solved by day i. We want to show there exists $j > i$ such that $a_j - a_i = 20$ since then you'll have solved exactly 20 problems on day $i+1$ through j.

   Note that $a_1 \geq 1$ since you solve at least one problem on day 1. Note also that $a_i < a_{i+1}$ for all i since you solve at least one problem every day. Finally, note that after k weeks, i.e., after $7k$ days, $7k \leq a_{7k} \leq 11k$.

   Let $b_i = a_i + 20$. If I can find $j > i$ such that $b_j = a_i$, I will be done since $b_j = a_j + 20 = a_i$.

   Note now that by definition, $b_1 \geq 21$, $b_i < b_{i+1}$ and $7k + 20 \leq b_{7k} \leq 11k + 20$.

   So we know all a_i's are distinct and we also know that all b_i's are distinct. If all a_i's were also distinct from the b_i's, then we would have $14k$ different values that come from $\{1, 2, \ldots, 11k + 20\}$. But that's only possible if $14k \leq 11k + 20$, i.e., when $k \leq \frac{20}{3}$. Since a year contains more than 7 weeks, we are guaranteed to find a period of consecutive days when you solve exactly 20 problems.

3. A positive integer is said to be *demonic* if it is written solely using 6's. For instance, 6, 66, 666 are all demonic. Prove that there exists a demonic number divisible by 2017.

   Note that there are infinitely many demonic numbers, but there are finitely many remainders of 2017 (2017 of them to be exact). By the pigeonhole principle, there must be two demonic numbers that have the same remainder. Suppose the demonic number, say α, made of a 6's and the demonic number made of b 6's, say β, where $b > a$ have the same remainder when divided by 2017, say r. Then $\beta - \alpha = \gamma \cdot 10^a \equiv 0($ mod 2017) where γ is the demonic number with $b - a$ 6's. That means that 2017 divides $\gamma \cdot 10^a$. Since 2017 is prime and $10^a = 2^a 5^a$, 2017 must divide γ. Thus there does exist a demonic number divisible by 2017.

4. Let S be any set of ten distinct integers chosen from $\{1, 2, \ldots, 99\}$. Show that S always contains two disjoint subsets S_1 and S_2 such that $\sum_{i \in S_1} i = \sum_{i \in S_2} i$.

There are $2^{10} = 1024$ different subsets of S. The sum of the elements of each subset is between 0 (if we have the empty subset) and $99+98+97+96+95+94+93+92+91+90 = 945$. So the sum of the elements in each of the 1024 different subsets must take one of 946 values, therefore two subsets must have that the sum of their elements are equal by the pigeonhole principle.