# Worksheet 2.1, 2.2 - Math 455

1. The vowels are a, e, i, o, u.
   (a) How many eleven-letter sequences from the alphabet contain exactly three vowels?
   (b) How many of these have at least one repeated letter?
2. Compute the number of ways to deal each of the following 5-card hand in poker. Note that the jack has value 11, queen 12, king 13, and an ace can have either value 1 or 14.
   (a) Straight (the values of the cards form a sequence of consecutive integers and do not have all of the same suit)
   (b) Flush (all five cards have the same suite, but they do not form a straight)
   (c) Straight flush (the cards form a sequence of consecutive integers and all have the same suit)
   (d) Four of a kind (four of cards have the same value)
   (e) Three of a kind (three of the cards have the same value and the other two cards have different values)
   (f) Full house (a pair and a three of a kind—incidentally, the name of the show that taught me English)
   (g) Two distinct matching pairs (but not a full house)
   (h) Exactly one matching pair (but no three of a kind)
   (i) At least one card from each suit.
   (j) At least one card from each suit, but no two values matching.
   (k) Three cards of one suit, and the other two of another suit, like three hearts and two spades.
3. Suppose in some lottery game, one selects six numbers between 1 and 2n. What fraction of all lottery tickets have the property that half the numbers are odd and half are even?
4. Assume that a positive integer cannot have 0 as its leading digit.
   (a) How many five-digit positive integers have no repeated digits at all?
   (b) How many have no consecutive repeated digits?
   (c) How many have at least one run of consecutive repeated digits (for example 23324, 45551, or 151155, but not 121212)?
5. (a) You decide to go riding a century with your bike for the first time. Afraid that you will starve, you decide to bring not one, not two, but THREE Clif bars, one in each of your jersey back pockets. You like variety, so you want to bring along three different flavors of Clif bars among the 19 equally delicious flavors that exist. What is the probability that you bring a brownie bar, a macadamia bar and a peanut butter bar if the likelihood of picking any flavor is equal? Explain carefully every part of the formula that you come up with.
   (b) You will first eat the Clif bar in your left pocket, then the one in your middle pocket and finish with the one in your right pocket. Being a Clif bar connoisseur, you actually feel that eating first a brownie bar, then a macadamia bar and finally a peanut butter bar has nothing to do with first eating a brownie bar, then a peanut butter bar and finally a macadamia bar. As a Clif bar snob, how many different Clif bar tasting menus can you create for your ride? Again, explain in minute detail every part of the formula that you come up with.

(c) As much as you love Clif bars, you must admit that you also like Kind bars and that they somehow feel a bit healthier. For your upcoming usual 50 mile ride, you know that a Clif bar and a Kind bar will suffice. Given that there are 29 different varieties of Kind bars, how many Clif bar/Kind bar duos exist? Again, explain your answer fully.

(d) An even hungrier friend decides to go on the century with you and wants to bring 4 bars, Kind or Clif. This person being much more normal than you, they don't care whether some of the bars are the same (or even all of the same) nor how many are Clif and how many are Kind. How many different assortments of bars are there for this philistine? Your answer should be as whole as a whole wheat bar—which neither Clif or Kind offers, that would be gross.

(e) Despite being abashed by the lack of discernment of your friend when it comes to snack bars, you appreciate the fact that they are willing to go ride 100km with you—and mostly that they put up with you, period. You offer to bring three Clif bars for them, and strongly advise them to buy a Kind bar as their final bar to have a balanced bar offering. In this case, how many different assortments of bars exist for your friend? Be at least as clear as the translucent packaging of Kind bars.

6. Use a combinatorial argument to prove that there are exactly $2^n$ different subsets of a set of $n$ elements. (Do not use the binomial theorem.)

7. Use algebraic methods to prove the cancellation identity: if $n$ and $k$ are non-negative integers and $m$ is an integer with $m \leq n$, then

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}.$$

8. You go bikecamping. Out of the $n$ different Clif bars you have at home, you bring $k$ for your trip. While on the trip, you select $m$ to bring on a hike. Show how you can count the number of possible combinations in two ways so that the cancellation identity of the previous problem appears.

9. Use induction to show that $\sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}$ if $m$ and $n$ are non-negative integers.