Worksheet 2.1, 2.2 - Math 455

Note that there are many, many ways to arrive to the same answer for these questions. If you got the same number though a different thought process, it is probably right!

1. The vowels are a, e, i, o, u.

   (a) How many eleven-letter sequences from the alphabet contain exactly three vowels?

   There are $\binom{11}{3}$ ways of choosing where the vowels will be in the eleven-letter sequence. Then for each of the 8 consonants in the sequence, there will be 21 choices, and for each of the 3 consonants in the sequence, there will be 5 choices. So $\binom{11}{3} \cdot 21^8 \cdot 5^3 \approx 7.8 \cdot 10^14$.

   (b) How many of these have at least one repeated letter?

   If all letters were different, then I would choose 8 consonants out of 21, and 3 vowels out of 5, and then I would arrange these 11 letters in whatever order I want. So there are $\binom{21}{8} \cdot \binom{5}{3} \cdot 11!$ different sequences with 3 vowels and 8 consonants where no letters are repeated. Thus, there are $\binom{11}{3} \cdot 21^8 \cdot 5^3 - \binom{21}{8} \cdot \binom{5}{3} \cdot 11! \approx 6.99 \cdot 10^14$ 11-letter sequences with exactly three vowels with at least one repeated letter.

2. Compute the number of ways to deal each of the following 5-card hand in poker. Note that the jack has value 11, queen 12, king 13, and an ace can have either value 1 or 14.

   (a) Straight (the values of the cards form a sequence of consecutive integers and do not have all of the same suit)

   A straight can start with 10 different card values (ace, two, ..., 10)—it cannot start with a jack or higher since then there are not enough bigger cards to have a sequence of five cards. Fixing the value of the lowest card fixes the value of the other cards in the straight, so all that is left to choose is the suit for each of the five cards in the hand. Each can take any of four suits. However, we do not want a straight flush. For a straight starting with a certain lowest card, there are four ways that it could be a straight flush: it could be all hearts, all diamonds, all clubs or all spades. Thus we get $10 \cdot 4^5 - 10 \cdot 4 = 10,200$.
   king the other a single). For the single card, we must also decide which one of the 4 suits it is. (For the quadruple, we must take all four suits out of four.) So we get $\binom{13}{2} \cdot \binom{2}{1} \cdot \binom{4}{1}\binom{4}{4} = 624$.

   (b) Three of a kind (three of the cards have the same value and the other two cards have different values)

   In such a hand, there will be a total of three different card values out of 13. We must choose which one of these three values is a triple (thus automatically making the other two singles). For the triple, we must choose 3 suits out of 4. For each of the single cards, we must choose which one of the 4 suits it is. So we get $\binom{13}{3} \cdot \binom{3}{1} \cdot \binom{4}{3} \cdot \binom{4}{1} \cdot \binom{4}{1} = 54,912$.

   (c) Full house (a pair and a three of a kind—incidentally, the name of the show that taught me English)

   In such a hand, there will be a total of two different card values out of 13. We must choose which one of these two values is a triple (thus automatically making the other a double). For the triple, we must choose 3 suits out of 4. For the double, we must choose 2 suits out of 4. So we get $\binom{13}{2} \cdot \binom{2}{1} \cdot \binom{4}{3} \cdot \binom{4}{2} = 3,744$.

(d) Two distinct matching pairs (but not a full house)

In such a hand, there will be a total of three different card values out of 13. We must choose which two of these three values are doubles (thus automatically making the other a single). For each double, we must choose 2 suits out of 4. For the single, we must choose 1 suit out of 4. So we get $\binom{13}{3} \cdot \binom{3}{1} \cdot \binom{4}{2}\binom{4}{2}\binom{4}{1} = 123,552$

(e) Exactly one matching pair (but no three of a kind)

In such a hand, there will be a total of four different card values out of 13. We must choose which one of these four values is a double (thus automatically making the others singles). For the double, we must choose 2 suits out of 4. For each single, we must choose 1 suit out of 4. So we get $\binom{13}{4} \cdot \binom{4}{1}\binom{4}{2}\binom{4}{1}\binom{4}{1}\binom{4}{1} = 1,098,240$.

(f) At least one card from each suit.

In such a hand, there will be one suit that there will be two cards from, and all the other suits will appear once. So we must choose which one of four suits appears twice. In that suit, we must choose two card values out of 13. In each of the other suits, we must choose 1 card value out of 13. So we get $\binom{4}{1}\binom{13}{2}\binom{13}{1}\binom{13}{1}\binom{13}{1} = 685,464$.

(g) At least one card from each suit, but no two values matching.

In such a hand, there will be one suit that there will be two cards from, and all the other suits will appear once. So we much choose which one of four suits appears twice. In that suit, we much choose two card values out of 13. In the next suit, we must choose 1 card out of the 11 values remaining. In the next one, 1 out of the 10 values remaining. And in the final suit, we must choose 1 card out of the 9 values remaining. So we get $\binom{4}{1}\binom{13}{2}\binom{11}{1}\binom{10}{1}\binom{9}{1} = 308,880$.

(h) Three cards of one suit, and the other two of another suit, like three hearts and two spades.

In such a hand, there will be two suits out of four. We must then choose which one of these two suits has three cards (thus automatically making the other suit have two cards). In the suit that contains three cards, we must choose 3 card values out of 13, and in the suit with two cards, we must choose 2 card values out of 13. We thus get $\binom{4}{2}\binom{2}{1}\binom{13}{3}\binom{13}{2} = 267,696$.

3. Suppose in some lottery game, one selects six numbers between 1 and 2n. What fraction of all lottery tickets have the property that half the numbers are odd and half are even?

We're assuming that the order doesn't matter in this lottery. Then one must choose 3 numbers out of n even numbers and 3 numbers out of n odd numbers. In total, one can choose any six numbers out of 2n. So we get

$$\frac{\binom{n}{3}\binom{n}{3}}{\binom{2n}{6}}.$$

4. Assume that a positive integer cannot have 0 as its leading digit.

(a) How many five-digit positive integers have no repeated digits at all?

There are nine choices for the first digit since zero cannot be it. Then we must choose which four of the remaining nine digits (i.e., different from the first one) we will use, and then choose in which of 4! ways to order them. So we get $9 \cdot \binom{9}{4} = 1,134$

(b) How many have no consecutive repeated digits?

There are nine choices for the first digit since zero cannot be it. Then for the next digit, it can be any digit but the one that we chose for the first digit, so there are nine choices. Similarly for the third, fourth and fifth digit. So we get $9^5 = 59,049$.

(c) How many have at least one run of consecutive repeated digits (for example 23324, 45551, or 151155, but not 121212)?

There are 90,000 five-digit positive integers (from 100,000 to 999,999, or if you prefer you have nine choices for the first digit and 10 for each subsequent digit). We know that 59,049 have no repeated digits whatsoever, so there are $90,000 - 59,049 = 30,951$.

5. (a) You decide to go riding a century with your bike for the first time. Afraid that you will starve, you decide to bring not one, not two, but THREE Clif bars, one in each of your jersey back pockets. You like variety, so you want to bring along three different flavors of Clif bars among the 19 equally delicious flavors that exist. What is the probability that you bring a brownie bar, a macadamia bar and a peanut butter bar if the likelihood of picking any flavor is equal? Explain carefully every part of the formula that you come up with.

First calculate how many different assortments you could bring. There are 19 choices for the first bar, 18 for the second (since you do not want to eat the same thing again) AND 17 for the third. However, you are overcounting each assortment 6 times because this counts each assortment in each possible order (brownie, macadamia, peanut butter and brownie, peanut butter, macadamia and ...), and there are 3! ways of ordering them. So in total, there are

$$\frac{19 \cdot 18 \cdot 17}{6} = \frac{19!}{16! \cdot 3!} = \frac{19!}{(19-3)!3!} = \binom{19}{3}$$

different assortments. Since any assortment is as likely as the next, the probability that you bring a brownie bar, a macadamia bar and a peanut butter bar is $\frac{1}{\binom{19}{3}}$.

(b) You will first eat the Clif bar in your left pocket, then the one in your middle pocket and finish with the one in your right pocket. Being a Clif bar connoisseur, you actually feel that eating first a brownie bar, then a macadamia bar and finally a peanut butter bar has nothing to do with first eating a brownie bar, then a peanut butter bar and finally a macadamia bar. As a Clif bar snob, how many different Clif bar tasting menus can you create for your ride? Again, explain in minute detail every part of the formula that you come up with.

Since order here matters, there truly is 19 choices for the first bar, 18 for the second AND 17 for the third. So there are

$$19 \cdot 18 \cdot 17 = \frac{19!}{16!} = \frac{19!}{(19-3)!} = 3! \cdot \binom{19}{3}$$

different tasting menus.

(c) As much as you love Clif bars, you must admit that you also like Kind bars and that they somehow feel a bit healthier. For your upcoming usual 50 mile ride, you know that a Clif bar and a Kind bar will suffice. Given that there are 29 different varieties of Kind bars, how many Clif bar/Kind bar duos exist? Again, explain your answer fully.

You want to buy any one of 19 Clif bars AND one of 29 Kind bars. So there are $19 \cdot 29$ duos.

(d) An even hungrier friend decides to go on the century with you and wants to bring 4 bars, Kind or Clif. This person being much more normal than you, they don't care whether some of the bars are the same (or even all of the same) nor how many are Clif and how many are Kind. How many different assortments of bars are there for this philistine? Your answer should be as whole as a whole wheat bar—which neither Clif or Kind offers, that would be gross.

This friend is willing to eat any of the 19+29=48 bars offered by both companies. So for each bar they eat, they have 48 choices: 48 for the first, 48 for the second, 48 for the third AND 48 for the fourth. So one might think that there are $48 \cdot 48 \cdot 48 \cdot 48 = 48^4 = 5308416$ choices for them. Unfortunately, by doing this, order matters and we are overcounting different assortment different numbers of times (for example, brownie, brownie, brownie, peanut butter is counted 4 times, whereas brownie, peanut butter, macadamia, mint chocolate is counted 4!=24 times.

Instead, let's think of how many duplicated bars they can bring: they can bring either 4 different bars, 2 of the same bars and two other different bars, 2 of the same bars and two other bars that are also the same, three of the same bars and 1 different bar OR 4 of the same bar.

If they bring 4 different bars, there are $\binom{48}{4}$ different assortments that exist.

If they bring 2 bars that are the same and then two extra bars that are different, there are $\binom{48}{3} \cdot 3$ different assortments since you can choose what three flavors to bring AND then decide which of the three flavors you duplicate. Alternatively, you can say you have 48 choices for the flavor that you have two of and then you must pick two extra flavors of the 47 flavors remaining, so $48 \cdot \binom{47}{2}$.

If they bring 2 of the same bars and two other bars that are the same (but different from the first two), there are $\binom{48}{2}$ different assortments since you choose two flavors out of the 48 to bring (and then you double each, so no choice has to be made further).

If they bring 3 of the same and one different bar, then there are $\binom{48}{2} \cdot 2$ assortments since you choose two flavors out of the 48 AND then decide which of the two you bring three of. Alternatively, you can say there are 48 flavors to choose from for the one you bring three of, AND then 47 flavors left for the extra bar, so $48 \cdot 47$.

Finally, if they bring 4 of the same bar, they must simply decide which of the 48 flavors to bring, so there are 48 choice.

So in total, there are

$$\binom{48}{4} + \binom{48}{3} \cdot 3 + \binom{48}{2} + \binom{48}{2} \cdot 2 + 48 = 249,900$$

assortments.

(e) Despite being abashed by the lack of discernment of your friend when it comes to snack bars, you appreciate the fact that they are willing to go ride 100km with you—and mostly that they put up with you, period. You offer to bring three Clif bars for them, and strongly advise them to buy a Kind bar as their final bar to have a balanced bar offering. In this case, how many different assortments of bars exist for your friend? Be at least as clear as the translucent packaging of Kind bars.

For the three Clif bars you bring for them, the situation is as in (d). Either you bring them three of the same, two of the same and one different OR three different Clif bars. If you bring them three of the same, there are 19 different assortments. If you bring them two of the same and one different, there are $\binom{19}{2} \cdot 2$ different assortments. Finally, if you bring them three different bars, there are $\binom{19}{3}$ different assortments. So in total, there are $19 + \binom{19}{2} \cdot 2 + \binom{19}{3} = 1330$ different assortments you bring. But we are not done: you bring them one of 1330 Clif bar assortments AND they bring any of 29 different Kind bars. So in total, there are

$$1330 \cdot 29 = 38570$$

different assortments.

6. Use a combinatorial argument to prove that there are exactly 2^n different subsets of a set of n elements. (Do not use the binomial theorem.)

For each element, you decide whether it is in the subset or not. So you have two choices for the first element AND two choices for the second element AND two choices for the third element AND ... AND two choices for the nth elemement. So we have 2^n different subsets of a set of n elements.

7. Use algebraic methods to prove the cancellation identity: if n and k are non-negative integers and m is an integer with $m \leq n$, then

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}.$$

Expanding the left-hand side, we get

$$\frac{n!}{k! \cdot (n-k)!} \cdot \frac{k!}{m!(k-m)!} = \frac{n!}{(n-k)!m!(k-m)!}.$$

Expanding the right-hand side, we get

$$\frac{n!}{m!(n-m)!} \cdot \frac{(n-m)!}{(k-m)!(n-m-(k-m))!} = \frac{n!}{m!(n-m)!} \cdot \frac{(n-m)!}{(k-m)!(n-k)!} = \frac{n!}{(n-k)!m!(k-m)!}.$$

8. You go bikecamping. Out of the n different Clif bars you have at home, you bring k for your trip. While on the trip, you select m to bring on a hike. Show how you can count the number of possible combinations in two ways so that the cancellation identity of the previous problem appears.

   You choose k out of the n bars you have at home to bring and then you choose m out of the k bars you brought to go hiking. So we get $\binom{n}{k}\binom{k}{m}$.

   Another way of getting the same is to choose which m bars you will bring on the hike, and then choose which $k-m$ out of the $n-m$ bars remaining to bring for the rest of your trip (so that you have k bars in total). So we get $\binom{n}{m}\binom{n-m}{k-m}$.

9. Use induction to show that $\sum_{k=0}^{n}\binom{k}{m} = \binom{n+1}{m+1}$ if m and n are non-negative integers.

   We proceed by induction on n. If $n = 0$, then we get $\binom{0}{m} = \binom{1}{m+1}$. If $m = 0$, then both sides are equal to 1. If m is anything else, then we get zero on both sides.

   Suppose it holds for n. We'll show it for $n+1$. First separate your sum by pulling out the term for $n+1$.

$$\sum_{k=0}^{n+1} = \binom{n+1}{m} + \sum_{k=0}^{n}\binom{k}{m} + \sum_{k=0}^{n}\binom{k}{m}.$$

   By the induction hypothesis, this is equal to

$$\binom{n+1}{m} + \binom{n+1}{m+1},$$

   which we have seen in class is equal to $\binom{n+2}{m+1}$ as desired.