Worksheet 1.7 - Math 455

1. Find the minimum size of a maximal matching in C_n.

   I cannot have a path of length three on C_n where none of these edges are in the matching. In that case, the matching would not be maximal since I could add the middle edge of that path and still have a matching. Therefore, since at least one of every three edges must be present, the minimum size of a maximal matching in C_n is $\lceil \frac{n}{3} \rceil$.

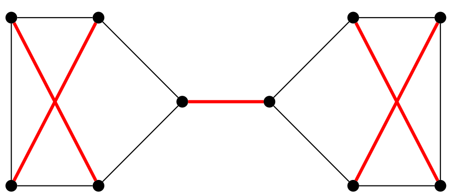
2. Let G be a bipartite graph. Show that G has a matching of size at least $\frac{|E(G)|}{\Delta(G)}$.

   Any vertex is adjacent to at most $\Delta(G)$ edges. So any edge cover must have size at least $\frac{|E(G)|}{\Delta(G)}$. By König's theorem, we know that the size of the mininmum edge cover is equal to the size of the maximum matching. Thus the maximum matching in the graph must have size at least $\frac{|E(G)|}{\Delta(G)}$.

3. Let k be some fixed integer between 1 and n. Let G be some subgraph of $K_{n,n}$ with more than $(k-1)n$ edges. Prove that G has a matching of size at least k.

   Note that any vertex is adjacent to at most n vertices since G is a subgraph of $K_{n,n}$. By the previous question, we thus get that there is a matching of size at least $\frac{(k-1)n+1}{n}$ since $|E(G)| \geq (k-1)n+1$. Thus, since this fraction is strictly greater than $k-1$, the max matching must have size at least k.

4. Draw a connected, 3-regular graph that has both a cut vertex and a perfect matching.

   This graph is 3-regular. The red edges form a perfect matching. The middle vertices are cut vertices—removing either disconnects the graphs.

5. Determine how many different perfect matchings there are in $K_{n,n}$.

   There are $n!$ since the first vertex in the part X can be matched to any of the n vertices in part Y, then the second vertex of part X can be matched to any of the remaining unmatched $n-1$ vertices in part Y, and so on.