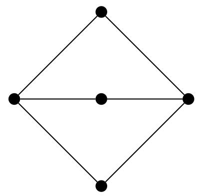
Worksheet 1.5 - Math 455

1. Find a planar representation of $K_{2,3}$.

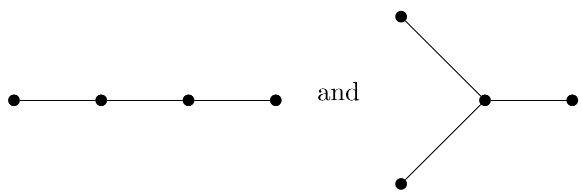


2. Draw a planar graph in which every vertex has degree exactly 5.

   See http://mathworld.wolfram.com/IcosahedralGraph.html

3. Let G_1 and G_2 be two planar graphs with n vertices, q edges, and r regions. Must they be isomorphic?

   No. Take for example two trees on n vertices that are not isomorphic. They both have n vertices, $n-1$ edges and 1 region.



4. How many regions are in a connected planar graph G of order 24 and regular degree 3?

   Such a graph has $\frac{24\cdot 3}{2} = 36$ edges. Since G is planar, we know by Euler's formula that the number of regions is $2 - 24 + 36 = 14$.

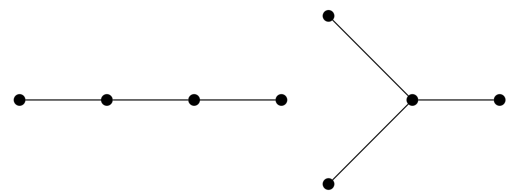
5. Let G be a connected planar graph of order n where $n < 12$. Prove that $\delta(G) \leq 4$.

   If $n \leq 5$, then it is trivial since each vertex has at most 4 neighbors.

   Suppose $\delta(G) \geq 5$ and that $6 \leq n \leq 11$. Then we obtain that $5n \leq \sum_{v \in V(G)} \deg(v)$ since each degree is at least 5. Furthermore, $\sum_{v\in V(G)} \deg(v) = 2 \cdot |E(G)| \leq 2(3n-6) = 6n - 12$ since G is planar. So $5n \leq 6n - 12$. Finally, note that $-12 < n$ since $n \leq 11$, so $5n \leq 6n - 12 < 5n$, a contradiction. So if $6 \leq n \leq 11$, then $\delta(G) \leq 4$ for connected planar graphs.

6. Prove that Euler's formula fails for disconnected graph.

   Take for example the following forest.



Here, $n = 8$, $q = 6$ and $r = 1$, but $8 - 6 + 1 \neq 2$.

7. Let G be of order $n \geq 11$. Show that at least one of G and $\bar{G}$ is nonplanar.

   First note that $|E(G)| + |E(\bar{G})| = \frac{n(n-1)}{2}$. If G and $\bar{G}$ were both nonplanar, then $E(G) \leq 3n - 6$ and $E(\bar{G}) \leq 3n - 6$. Therefore, $\frac{n(n-1)}{2} \leq 6n - 12$ which is equivalent to saying $0 \leq -\frac{1}{2}n^2 + \frac{13}{2}n - 12$. Plotting this quadratic function, we see it lies above the y-axis when $n \in \{3, 4, \ldots, 10\}$. So if $n \geq 11$, both graphs cannot be planar.

8. Show that there is no polyhedron with 5 vertices such that each pair of vertices is connected by an edge.

   The skeleton of such a graph would be K_5, which we know is nonplanar. But the skeleton of any polyhedron is a planar graph. Thus there cannot be such a polyhedron.

9. For a regular tetrahedron, take the midpoint of each of the 6 edges. Show that the solid whose vertices are those points is a regular octahedron.

   Consider the regular cube with the vertices $(\pm 1, \pm 1, \pm 1)$. Take every other vertex, that is, $(1, 1, 1)$, $(-1, -1, 1)$, $(-1, 1, -1)$ and $(1, -1, -1)$. Notice that the distance between each of these points is $\sqrt{8}$, and so they are the vertices of a regular tetrahedron. Now the midpoints of each edge are the six points $(\pm 1, 0, 0)$,$(0, \pm 1, 0)$, and $(0, 0, \pm 1)$, which are the vertices of a regular octahedron with sidelength $\sqrt{2}$.

10. Let F_k be the number of faces of a polyhedron P that are k-gons. For a simple polyhedron, i.e., a polyhedron where every vertex has degree 3, show that $3F_3 + 2F_4 + F_5 - F_7 - 2F_8 - 3F_9 - \ldots \geq 12$.

    Sorry, I meant to put an easier question! The idea would be to first show that $\sum_{n \geq 3}(6-n) = 4E - 6V + 12$ and then use the fact that $2E \geq 3V$, and then to look at the dual polytope.