Worksheet 1.4 - Math 455

1. Draw an Eulerian graph that satisfies the following conditions, or prove that no such graph exists.
   (a) An even number of vertices, an even number of edges.
   (b) An even number of vertices, an odd number of edges.
   (c) An odd number of vertices, an even number of edges.
   (d) An odd number of vertices, an odd number of edges.
2. Show that a connected graph G contains an Eulerian trail if and only if there are zero or two vertices of odd degree.
3. Let G be a connected graph which is regular of degree $r \geq 1$. Prove that the line graph of G, denoted $L(G)$, is Eulerian. (The line graph $L(G)$ of a graph G is defined as follows. The vertices of $L(G)$ are the edges of G, and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G share a vertex.)
4. Let $G = K_{n_1,n_2}$. Find conditions that characterize when
   (a) G will have an Eulerian trail,
   (b) G will be Eulerian.
5. Show that if G is Hamiltonian, then G is 2-connected.
6. Is the independence number of a bipartite graph equal to the cardinality of one of its partite sets? Why or why not?
7. Show that if G has n vertices and is regular of degree $r \geq 1$, then $\alpha(G) \leq \frac{n}{2}$.
8. Show that the line graph $L(G)$ of any graph G is claw-free.

Hints:

1. Yes, yes, yes, yes.
2. What vertices on your Eulerian trail can have odd degree?
3. What will be the degree of every vertex in $L(G)$ in terms of r?
4. How can K_{n_1,n_2} have 0 or two vertices with odd degree?
5. How many paths are there at least between any two vertices?
6. What if the graph is not connected?
7. Consider the vertices in a maximum independent set S. Any vertex not in it must form at least one edge with some vertex of S—with at most how many vertices of S can it form an edge?
8. Consider the vertex of degree 3 in your claw. In G, where was that vertex and how were the other vertices of your claw?