Worksheet 1.4 - Math 455

1. Draw an Eulerian graph that satisfies the following conditions, or prove that no such graph exists.
   (a) An even number of vertices, an even number of edges.

       Any even cycle will do.
   (b) An even number of vertices, an odd number of edges.
       For example, two triangles glued together by an edge.
   (c) An odd number of vertices, an even number of edges.
       For example two triangles glued together by a vertex.
   (d) An odd number of vertices, an odd number of edges.
       Any odd cycle will do.

2. Show that a connected graph G contains an Eulerian trail if and only if there are zero or two vertices of odd degree.

   $\Rightarrow$: Only the first and last vertex of an Eulerian trail can have odd degree. Every time any other vertex is visited, you will come in and out through different edges, and since no edges repeat, the degree of such vertices must be even. The first and last vertices if and whenever they are visited at other times will be visited in the same fashion. So if the first and last vertex are different, their degree is an even number plus one, so an odd number. If they are the same, i.e., if our trail is actually a circuit, then every vertex will have an even degree.

   $\Leftarrow$: If there are zero vertices with odd degree, then by the theorem we saw in class, there is an Eulerian circuit which is also a trail. If there are two vertices with odd degree, say u and v, let P be a path from u to v (this exists since G is connected). Remove the edges of P from G. Then every vertex has even degree, and so there exists an Eulerian circuit C in $G - E(P)$. Note that C must share a vertex, say w, with P otherwise the graph would not be connected. In G, go from u to w on P, then visit C which brings you back to w, and from there on continue on P to v. This is an Eulerian trail.

3. Let G be a connected graph which is regular of degree $r \geq 1$. Prove that the line graph of G, denoted $L(G)$, is Eulerian. (The line graph $L(G)$ of a graph G is defined as follows. The vertices of $L(G)$ are the edges of G, and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G share a vertex.)

   In G each edge $e = \{u, v\}$ is adjacent to $r - 1$ edges at u and to $r - 1$ other edges at v. Thus, in $L(G)$, each vertex will have degree $2(r - 1)$.

4. Let $G = K_{n_1,n_2}$. Find conditions that characterize when
   (a) G will be Eulerian,
       G will be Eulerian if and only if every vertex has even degree, which is the case if and only if both n_1 and n_2 are even.

(b) G will have an Eulerian trail. If the trail is actually a circuit, then the answer is above. Otherwise G will have an Eulerian trail (that is not a circuit) if and only if it has exactly two vertices with odd degree. One option is $K_{1,1} = K_2$. Otherwise, $K_{2,m}$ where m is odd will have exactly two odd degree vertices and the other vertices will have even degree.

5. Show that if G is Hamiltonian, then G is 2-connected.

   Since G is Hamiltonian, there is a cycle that includes every vertex in G, and so there is at least two paths between any two vertices of G.

6. Is the independence number of a bipartite graph equal to the cardinality of one of its partite sets? Why or why not?

   No. Take for example four vertices with no edges. You can say it is a bipartite graph $G = (X \cup Y, E)$ by putting two vertices in X and putting the other two vertices in Y. The independence number is 4 but the cardinality of each partite set is 2.

7. Show that if G has n vertices and is regular of degree $r \geq 1$, then $\alpha(G) \leq \frac{n}{2}$.

   Let S be a maximum independent set in G, i.e., $|S| = \alpha(G)$. Note that all edges adjacent to some vertex in S must be going into $V(G) \backslash V(S)$ since S contains no edge. Since every vertex has degree r, the number of edges coming out of S is exactly $r \cdot \alpha(G)$. Thus $E(G) \geq r \cdot \alpha(G)$. We also know that $E(G) = \frac{1}{2} \sum_{v \in V(G)} \deg(v) = \frac{1}{2} \cdot n \cdot r$ since G is r-regular. Thus $\frac{1}{2} \cdot n \cdot r \geq r \cdot \alpha(G)$ which implies that $\frac{1}{2} n \geq \alpha(G)$.

8. Show that the line graph $L(G)$ of any graph G is claw-free.

   Consider any vertex w of degree $d \geq 3$ in $L(G)$ and suppose this is the middle vertex of a possible claw. In G, this corresponds to an edge $e = \{u, v\}$ that is adjacent to exactly d other edges. Since $d \geq 3$, at least u or v is adjacent to at least 2 of these d edges (otherwise, if u and v are both adjacent to at most one of these edges, then w in $L(G)$ would have degree at most 2). Note that these two other edges correspond to two vertices in $L(G)$ that form an edge. Therefore, if you select any three vertices adjacent to w in $L(G)$, at least two of them will form an edge, and we can't have a claw.