Worksheet 1.3 - Math 455

1. Show that every nonleaf in a tree is a cut vertex.

   A nonleaf has degree greater or equal to two by definition. Take any two vertices, say a and b, adjacent to a nonleaf, say v. If there exists a path P between a and b in $G - v$, then P with the edges $\{a, v\}$ and $\{v, b\}$ form a cycle in the tree, a contradiction. Thus, a and b must be in different connected components of $G - v$, meaning that v was a cut vertex.

2. What is the connectivity of G if G is a tree of order at least 2?

   Any tree of order greater or equal to 3 contains a nonleaf, so by the previous question, the connectivity is 1. The only tree of order 2 is K_2 which also have a connectivity of one.

3. Draw all unlabeled trees of order 7.

   See for example https://math.stackexchange.com/questions/407562/gallery-of-unlabelled-trees-with-n-vertices

4. Count how many unlabeled forests of order 6 exist.

   Using the trees of order smaller or equal to six in the book, we have
   6 forests with one tree of order 6
   3 forests with one tree of order 5 and one tree of order 1
   2 forests with one tree of order 4 and one tree of order 2
   2 forests with one tree of order 4 and two trees of order 1
   1 forest with two trees of order 3
   1 forest with one tree of order 3, one tree of order 2 and one tree of order 1
   1 forest with one tree of order 3 and three trees of order 1
   1 forest with three trees of order 2
   1 forest with two trees of order 2 and two trees of order 1
   1 forest with one tree of order 1 and four trees of order 1
   1 forest with six trees of order 1
   for a total of 20.

5. Prove that all trees of order at least two are bipartite graphs.

   By the theorem that a graph is bipartite if and only if it contains no odd cycle and by the definition of a tree being a connected acyclic graph, the result holds.

6. How many paths are there between any two vertices in a tree?

   Exactly one. Since G is connected, we know there must be at least one path between any two vertices. Since there are no cycles, there cannot be two paths between any two given vertices.

7. Show that a forest on n vertices with k connected components contains $n-k$ edges.

   Suppose the k components have order $n_1, \ldots, n_k$. Note that $n_1 + \ldots + n_k = n$. Each component is a tree and thus has $n_i - 1$ edges. Thus, the total number of edges is

$$\sum_{i=1}^{k}(n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = n - k.$$

8. Show that a graph of order n is a tree if and only if it is acyclic and contains $n-1$ edges.

   "$\Rightarrow$": If G is a tree, then it is acyclic by definition and it has $n-1$ edges by the theorem we saw in class.
   "$\Leftarrow$": If G is acyclic, then it is a forest. To show that it is a tree, we need to show that G is connected. By question 7, we know the number of edges in a forest is $n-k$ where k is the number of connected components. Since G has $n-1$ edges, G must have only one connected component, and thus G is connected.

9. Show that any tree with an even number of edges has at least one vertex with even degree.

   Suppose not: suppose the degree of every vertex is odd. Since the number of edges is one less than the number of vertices, we know the number of vertices is odd. Adding up the degrees of all the vertices gives us an odd number, but we know that the sum of the degrees gives us twice the number of edges, an even number. This is a contradiction.

10. Show that every connected graph contains at least one spanning tree.

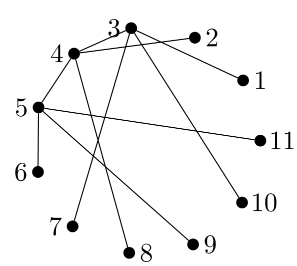
    We showed that Kruskal's algorithm returns the minimum weight spanning tree in any graph, so a spanning tree must exist.

11. Let G be connected, and let e be an edge of G. Prove that e is a bridge if and only if it is in every spanning tree of G.

    "$\Rightarrow$" If e is a bridge, then $G - e$ consists of two connected components. A spanning tree that doesn't contain e would have to be contained in $G - e$ and could thus not be connected, a contradiction to the definition of a tree.
    "$\Leftarrow$" Suppose e is in every spanning tree of G. Note that $G - e$ is also a graph on n vertices. If it were connected, then by question 10, we could find a spanning tree in it, and that spanning tree would not contain e. Thus $G - e$ cannot be connected, so e must be a bridge.

12. Give an example of a connected, weighted graph G having a cycle with two identical weights, which is neither the smallest nor the largest weight in the graph, and a unique minimum weight spanning tree which contains exactly one of these two identical weights.

13. Draw and label a tree whose Prüfer sequence is 5,4,3,5,4,3,5,4,3.



14. Let T be a labeled tree. Prove that the Prüfer sequence of T will not contain any of the leaves' labels.

    The algorithm to build the Prüfer sequence records the neighbor of the smallest leaf in the current tree and then deletes the leaf. The only way a leaf in the original tree could be recorded is if the vertex adjacent to it (there is only one since it is a leaf) is the smallest leaf left in the tree at some point. But in that case, we have two leaves adjacent to each other, so the tree has become K_2 and so the algorithm stops.

15. Show that every vertex v of a labeled tree T appears in the Prüfer sequence of T exactly $\deg(v) - 1$ times.

    All but two vertices in T eventually disappear throughout the algorithm. For a vertex v to disappear, it needs to become a leaf, i.e., all but one of the vertices adjacent to it must get removed, and when they do, v gets recorded each time. Thus v gets recorded $\deg(v) - 1$ times.

    If a vertex v doesn't disappear, it is a vertex of K_2 at the end of the algorithm. It thus also has degree one and has also been recorded $\deg(v) - 1$ times.