Worksheet 1.2 - Math 455

1. Show that if A is the adjacency matrix of some graph G, then $[A^k]_{i,j} = 0$ for all $1 \leq k < d(v_i, v_j)$.

   Since $[A^k]_{i,j}$ is the number of i,j-walks of length k, if k is smaller than the length of the shortest path between i and j, then no i,j-walk of length k exists (since we've already proven that any i,j-walk contains a path from i to j), and so $[A^k]_{i,j} = 0$.

2. Find A^3 where A is the adjacency matrix of K_4 without computing it directly.

   The number of walks of length 3 from a vertex to itself is the same for every vertex in K_4 since all vertices look the same. The only way I can do it is to visit two other distinct vertices and come back to the original vertex. So, starting at vertex i, I need to choose which two out of three other vertices I'll visit, say j and k, and in which order I will visit them (j then k or k then j). So the total number of walks of length 3 from a vertex to itself is $\binom{3}{2} \cdot 2 = 6$.

   Similarly, the number of walks of length 3 from a vertex to a different vertex is the same for every pair of vertices in K_4. So suppose I want to go from vertex i to j in three steps. One of three things can happen: I can first visit any other vertex besides i (including j), come back to i, and then go to j (there are three ways of doing that), or I go straight to j, and then visit some other vertex k (not i since we've already counted that case) and come back to j (there are thus two ways of doing that), or I visit the remaining two vertices in whichever order before going to j (there are also two ways of doing that. So in total there are $3 + 2 + 2 = 7$ walks of length 3 from i to j.

   Therefore $A^3 = \begin{pmatrix} 6 & 7 & 7 & 7 \\ 7 & 6 & 7 & 7 \\ 7 & 7 & 6 & 7 \\ 7 & 7 & 7 & 6 \end{pmatrix}$

3. If A is the adjacency matrix of some graph G, show that $[A^2]_{j,j} = \deg(v_j)$.

   $[A^2]_{j,j}$ is the jth row of A dotted with the jth column of A. First note that these two vectors are the same since A is symmetric. Thus, the dot product is equal to summing up the squares of each entry of either of these vectors. Since each entry is a zero or a one, and since $0^2 = 0$ and $1^2 = 1$, this particular dot product is equivalent to summing up the entries in either vector or simply counting the number of ones in either vector. Since the number of ones is equal to the number of vertices i such that $\{i, j\} \in E(G)$ by the definition of A, the total is the number of vertices adjacent to j, i.e., the degree of j.