Worksheet 1.1 - Math 455

1. Let $G = (V, E)$. Show that $|E| = \frac{1}{2}\sum_{v\in V} \deg(v)$.

   Adding up the degrees of all the vertices, you end up counting each edge twice, once for each of its endpoints. Thus $\sum_{v\in V} \deg(v) = 2|E|$.

2. Show that the number of vertices with odd degree in any graph is even.

   By the previous question, we know that summing up the degrees should give us an even number. This means that the number of vertices with odd degrees need to be even so that the sum can be even.

3. Let $G = (V, E)$ where $|V| \geq 2$. Show that the degree sequence has at least one pair of repeated entries.

   Suppose there exists a connected component that contains k vertices where $k \geq 2$. The degree of every vertex in that component is between 1 and $k - 1$. Since there are k vertices, two of them must have the same degree.

   If there doesn't exist a connected component containing at least two vertices, then my graph is made of isolated vertices, and so every degree is the same: zero.

4. Show that any walk between any two given vertices in a graph contains a path between those two vertices.

   We proceed by induction on the length of the walk. If there is a walk of length one between vertices u and v, then $\{u, v\} \in E(G)$ and that edge is also a path between u and v.

   Let's now assume that any walk of length less or equal to $k - 1$ between vertices u and v also contains a path between u and v, and let's prove that the same holds for a walk of length k.

   If the walk of length k between vertices u and v contains no repeated vertices, then it is a path, and we are done.

   Otherwise, some vertex gets visited multiple times. Let w be the first vertex that you visit in your u, v-walk that you eventually visit again. Certainly, stopping your walk at the first time you visit w gives you a u, w-path. Removing these vertices from your walk leaves you with a w, v-walk of length less than k. By the induction hypothesis, this w, v-walk contains a w, v-path. Gluing this w, v-path after the u, w-path you found gives you a u, v-path. Indeed, besides w, there cannot be any common vertices in those two paths since w was the first vertex that ever got visited again in the walk.

5. Show that every finite graph having exactly two vertices of odd degree must contain a path from one to the other.

   To count the number of edges in a connected component, one can add up the degrees of the vertices in the component and divide by two (for the same reason as in problem 1). Just as in problem 2, this implies that the number of odd degree vertices in a connected component must be even. Thus, if there are exactly two odd degree vertices in a graph, they must be in the same connected component. By the definition of a connected component, there is a path between any two vertices in the component, including between these two odd degree vertices.

6. Show that every closed odd walk contains an odd cycle.

   We proceed again by induction on the length of the closed odd walk.

   It is true if the length is 1: then we have a loop which is also a cycle of length 1 (assuming we're allowing non-simple graphs—otherwise, you'll want to consider the case where the length is 3 and show that in that case, you can't visit the same vertex twice except to close the walk and thus have an odd cycle).

Suppose it is true for all closed odd walks of length less or equal to $2k-1$, that is, all closed odd walks of length less or equal to $2k-1$ contain an odd cycle. We want to show this still holds for closed odd walks of length $2k+1$.

If the closed odd walk of length $2k+1$ contains no repeated vertices except for the first and last vertex, then it is a cycle of odd length, and we are done.

Otherwise, some other vertex must get repeated multiple times. Consider the vertex that is the first to be encountered a second time along your walk, say w. Between the moment when you first meet w and when you meet it a second time, no other vertex gets repeated, so the walk between your first and second encounter with w is a cycle. Either that cycle has odd length, and we are done, or that cycle has even length. In that case, remove that cycle from your walk—you are left with a closed walk, and it's length is odd since you have removed an even number of edges, and its length is also at most $2k-1$ since any even cycle has length at least 2. Thus, by the induction hypothesis, I know that this closed odd walk contains an odd cycle.

7. Let $G = (V, E)$ such that $|V| = n$ and $|E| < n-1$. Show that G is not connected.

   First draw your n vertices. Note that at this point, your graph has n connected components. Then start adding your edges one by one. Either both endpoints of the edge you're adding are in the same connected component, and then the number of connected components remains the same. Or they are in different connected components, and adding that edge turns these two connected components into a new bigger connected component, thus decreasing the total number of connected components by one. Since you are adding at most $n-2$ edges, the total number of connected components will decrease by at most $n-2$ as you add all of the edges, meaning that in the end, the total number of connected components will be at least 2. Thus G is not connected.

8. Show that every 2-connected graph contains at least one cycle.

   If the graph is K_n, the result is clear. Otherwise, pick two vertices u and v that are not adjacent. Since the graph is connected, you know there exists a path, say P_1, between any two vertices in the graph, including between u and v. Since u and v are not adjacent, this path has length at least two and contains at least one other vertex. Remove a vertex, say w, on this u, v-path from the graph. Since the graph is 2-connected, you know that $G - w$ is still connected, so there still exists some different path, say P_2, from u to v. Two cases arise.

   - **P_2 contains a vertex that is not in P_1.** Let x be the first vertex along P_2 that is not in P_1. Let x' be the vertex preceding x on path P_2; note that x' is in P_1 as well. Let y be the first vertex that is after x on path P_2 that is also on P_1 (at the latest, this will be vertex v). Following P_2 from x' to y and then P_1 from y back to x' forms a cycle.
   - **Every vertex in P_2 is in P_1.** Still, given that w is not in P_2, this means that we have some edge, say xy, that is in P_2 that isn't in P_1. Following P_1 from x to y and then following the edge xy back gives us a cycle since x and y are not adjacent in P_1.

9. Show that for every graph G, $\kappa(G) \leq \delta(G)$.

   Let v be a vertex with degree $\delta(v)$. Note that in $G - N(v)$, v is an isolated vertex, so $G - N(v)$ is disconnected, meaning that $N(v)$ is a cutset. This implies that $\kappa(G) \leq |N(v)| = \delta(G)$.

10. True or false? If G has no bridges, then G has exactly one cycle. Explain.

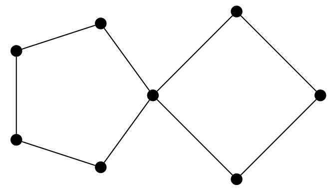
    False. K_n where $n \geq 4$ has no bridges, and more than one cycle.

11. True or false? If G has no cut vertices, then G has no bridges. Explain.

    False. K_2 has no cut vertex, but it has a bridge. This is an exception though: if a connected graph on three vertices or more contains a bridge, then it contains a cut vertex. Indeed, suppose $G - uv$ gives two connected components, say G_u (which contains vertex u) and G_v (which contains vertex v). If $|V(G_u)| \geq 2$, then $G_u - u$ is not empty. Therefore, u is a cut vertex since $G_u - u$ is not empty and is in a different component as v since uv was a bridge for G. Similarly, if $G_v - v$ is not empty, then v is a cut vertex. The only way both $G_u - u$ and $G_v - v$ can be empty is if G_u and G_v consist exactly of one vertex each. If G is connected, the only way this can be is if G is K_2.

12. True or false? If G has no bridges, then G has no cut vertices. Explain.

    False. Take for example

    which contains a cut vertex but no bridge.

13. If K_{r_1,r_2} is regular, prove that $r_1 = r_2$.

    I meant for this question to be a bit more challenging than that. I'll show the stronger statement that "If G is a regular bipartite graph with parts of size r_1 and r_2, then $r_1 = r_2$.

    Suppose the degree at every vertex is d. Then I can count the total number of edges by adding the degrees of the vertices in one part (that way, every edge is counted exactly one). This implies that the number of edges is $r_1 \cdot d$ if I add up the degrees of the vertices in the first part. If I do the same thing using the second part, I get that the number of edges is $r_2 \cdot d$. Thus $r_1 \cdot d = r_2 \cdot d$ which implies that $r_1 = r_2$.

14. The *complete multipartite graph* $K_{r_1,r_2,\ldots,r_k}$ is the graph that consists of k sets of vertices $A_1, A_2, \ldots, A_k$ (called *parts*), with $|A_i| = r_i$ for each I, where every pair of vertices from two different parts form an edge, and every pair of vertices coming from the same part do not form an edge. Find an expression for the order and size of $K_{r_1,r_2,\ldots,r_k}$.

    The order is the total number of vertices, so $\sum_{i=1}^{k} r_i$.

    The size is the total number of edges. Between parts A_i and A_j, there are $r_i \cdot r_j$ edges. I need to do this for every pair of parts. There are different ways in which one can write this; here is one.

    $$\sum_{1 \leq i < j \leq k} r_i \cdot r_j$$

15. Prove that if the graphs G and H are isomorphic, then their complements $\bar{G}$ and $\bar{H}$ are also isomorphic.

    If G and H are isomorphic, then there exists a one-to-one correspondence $f : V(G) \to V(H)$ such that for each pair of vertices x, y of G, $xy \in E(G)$ if and only if $f(x)f(y) \in E(H)$. This implies that $xy \notin E(G)$ (i.e., $xy \in E(\bar{G})$) if and only if $f(x)f(y) \notin E(H)$ (i.e., $f(x)f(y) \in E(\bar{H})$) for any pair of vertices x, y of G. Note that $V(G) = V(\bar{G})$ and $V(H) = V(\bar{H})$. So f is a one-to-one correspondence between $V(\bar{G})$ and $V(\bar{H})$ such that for every pair of vertices in $\bar{G}$, $xy \in E(\bar{G})$ if and only if $f(x)f(y) \in E(\bar{H})$. Thus, $\bar{G}$ and $\bar{H}$ are isomorphic.