Worksheet 1.3 - Math 455

1. Show that every nonleaf in a tree is a cut vertex.
2. What is the connectivity of G if G is a tree of order at least 2?
3. Draw all unlabeled trees of order 7.
4. Count how many unlabeled forests of order 6 exist.
5. Prove that all trees of order at least two are bipartite graphs.
6. How many paths are there between any two vertices in a tree?
7. Show that a forest on n vertices with k connected components contains $n - k$ edges.
8. Show that a graph of order n is a tree if and only if it is acyclic and contains $n - 1$ edges.
9. Show that any tree with an even number of edges has at least one vertex with even degree.
10. Show that every connected graph contains at least one spanning tree.
11. Let G be connected, and let e be an edge of G. Prove that e is a bridge if and only if it is in every spanning tree of G.
12. Give an example of a connected, weighted graph G having a cycle with two identical weights, which is neither the smallest nor the largest weight in the graph, and a unique minimum weight spanning tree which contains exactly one of these two identical weights.
13. Draw and label a tree whose Prüfer sequence is 5,4,3,5,4,3,5,4,3.
14. Let T be a labeled tree. Prove that the Prüfer sequence of T will not contain any of the leaves' labels.
15. Show that every vertex v of a labeled tree T appears in the Prüfer sequence of T exactly $\deg(v) - 1$ times.

Hints:

1. Take two vertices that are adjacent to a nonleaf (why do such vertices exist?). Show that there cannot be a path between them that doesn't go through that nonleaf.
2. Use the last question and make sure your answer makes sense for trees of order 2 as well.
3. It helps to build things systemically starting with trees of order 6 (which are listed in your book).
4. First decide how many trees a particular forest contains, and then decide the order of those trees (what should the total order be?). Then use the list of trees of order 6 and less in your book.
5. What theorem do you know about bipartite graphs?
6. What does the fact that a tree is a connected graph implies? What does the fact that it does not contain a cycle imply?
7. Use the fact that a tree on n vertices contains $n-1$ edges.
8. One direction is easy. For the other direction, use the last question.
9. Think of the double-counting argument for the number of edges.
10. Can Kruskal's algorithm fail?
11. If e is a bridge, can $G-e$ contain a spanning tree? If e is not a bridge, can $G-e$ contain a spanning tree? When it does, is that spanning tree also a spanning tree for G?
12. You'll need more than one cycle!
13. Use Prüfer's method for assigning a labeled tree to a sequence.
14. What does one record in Prüfer's method for assigning a sequence to a labeled tree?
15. To remove v from T, v must have become a leaf at this point of the algorithm. So what must have happened for this to occur?