Worksheet 1.1 - Math 455

1. Let $G = (V, E)$. Show that $|E| = \frac{1}{2}\sum_{v \in V} \deg(v)$.
2. Show that the number of vertices with odd degree in any graph is even.
3. Let $G = (V, E)$ where $|V| \geq 2$. Show that the degree sequence has at least one pair of repeated entries.
4. Show that any walk between any two given vertices in a graph contains a path between those two vertices.
5. Show that every finite graph having exactly two vertices of odd degree must contain a path from one to the other.
6. Show that every closed odd walk contains an odd cycle.
7. Let $G = (V, E)$ such that $|V| = n$ and $|E| < n - 1$. Show that G is not connected.
8. Show that every 2-connected graph contains at least one cycle.
9. Show that for every graph G, $\kappa(G) \leq \delta(G)$.
10. True or false? If G has no bridges, then G has exactly one cycle. Explain.
11. True or false? If G has no cut vertices, then G has no bridges. Explain.
12. True or false? If G has no bridges, then G has no cut vertices. Explain.
13. If K_{r_1,r_2} is regular, prove that $r_1 = r_2$.
14. The *complete multipartite graph* $K_{r_1,r_2,\ldots,r_k}$ is the graph that consists of k sets of vertices $A_1, A_2, \ldots, A_k$ (called *parts*), with $|A_i| = r_i$ for each I, where every pair of vertices from two different parts form an edge, and every pair of vertices coming from the same part do not form an edge. Find an expression for the order and size of $K_{r_1,r_2,\ldots,r_k}$.
15. Prove that if the graphs G and H are isomorphic, then their complements $\bar{G}$ and $\bar{H}$ are also isomorphic.

Hints

1. How many times do you count each edge in $\sum_{v \in V} \deg(v)$?
2. This is a consequence from the previous question.
3. What values can each degree take for the vertices within a connected component on k vertices?
4. Try using induction.
5. Can the two vertices with odd degree be in different connected components?
6. Try using induction on the length of the walk.
7. Start with n vertices. Try to think about what happens algorithmically when you build your graph edge by edge. What can happen to the number of connected components when you add an edge? How many connected components did you start with?
8. Take any two vertices: how many paths must there be between them? There is certainly at least one since the graph is connected. If that path has length at least two, what happens if you remove a vertex?
9. Can you think of $\delta(G)$ vertices that you can remove to obtain a disconnected graph?
10. Does there exist graphs with many cycles that have no bridges?
11. How many paths must there be between any two vertices in a graph with no cut vertices? Refer to the question 8.
12. Does there exist graphs with a cut vertex that have no bridges?
13. Suppose not and count the number of edges in two different ways.
14. Count the number of edges between any two parts.
15. Use the definition that states that G and H are isomorphic if there exists a one-to-one correspondence $f : V(G) \to V(H)$ such that for each pair of vertices x, y of G, $xy \in E(G)$ if and only if $f(x)f(y) \in E(H)$.