Homework 4 - Math 409

In preparation of Quiz 4 on April 25

1. Show that the dual of the dual is the primal.

2. Show that, if the primal is feasible but the dual is infeasible, then the primal will be unbounded.

3. (a) List all the faces of the the tetrahedron $P = \{x \in \mathbb{R}^3 | x_i \geq 0 \ \forall i \in [3], x_1 + x_2 + x_3 \leq 1\}$.

   (b) Let $Q = \{x \in \mathbb{R}^n | x_i \geq 0 \ \forall i \in [n], \sum_{i=1}^n x_i \leq 1\}$. How many faces does Q have? How many faces of dimension k does Q have?

4. Show that $P = \{x \in \mathbb{R}^n | Ax \leq b\}$ has no vertices if $\text{rank}(A) < n$.

5. Let G be any graph. Let P be the convex hull of all perfect matchings in G. Show that the vertices corresponding to matchings M_1 and M_2 are adjacent on P if and only if $M_1 \Delta M_2$ has exactly one connected component. To do so, proceed in two steps.

   First, show that if $M_1 \Delta M_2$ has more than one connected components, then you can build two new matchings M_3 and M_4 that contain together the same edges (with the same multiplicity) as M_1 and M_2 together. What does this imply about faces containing M_1 and M_2?

   Then, assume that $M_1 \Delta M_2$ has exactly one connected component. Note that two vertices are adjacent on P if and only if there exists an objective function c such that these two vertices are the only ones minimizing $c^\top x$ over P. Find such a function c for the vertices corresponding to M_1 and M_2 to show that they form an edge.