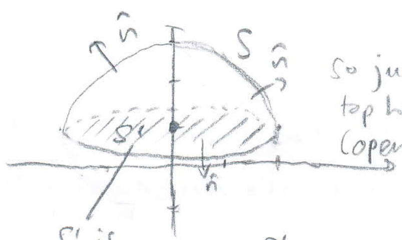
Annie's Survival Kit 8 - Math 324

1. (10 points) Let S be the part of the surface $x^2+y^2+(z-1)^2 = 4$ (oriented with $\hat{\mathbf{n}}$ pointing outwards) lying above the plane $z = 1$. Let $\mathbf{F} = \langle 4x^3 + x - \sin(yz), -12x^2y + e^{z^3}, x^2 + y^2 \rangle$. Find the flux through S using the divergence theorem. Recall that the divergence theorem states that $\int\int_S \mathbf{F} \cdot \hat{\mathbf{n}} dS = \int\int\int_R \operatorname{div}(\mathbf{F}) dV$ ($\nabla \cdot \vec{F}$) when $\mathbf{F}$ is defined and differentiable everywhere on S and R where S is a closed surface and R is the region that it contains. Note here that S is not closed.



So just S is top hemisphere (open bottom)

S' is the disk of radius 2 needed to close S

S': $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 1 \end{cases}$ $0 \le r \le 2$, $0 \le \theta \le 2\pi$

Note that $\vec{F}$ is defined and differentiable everywhere, so I can apply the divergence theorem, but to do so, I need to close my surface.

So: $\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_{S+S'} \vec{F}\cdot\hat{n}\,dS - \iint_{S'} \vec{F}\cdot\hat{n}\,dS$

$= \iiint_R \operatorname{div}(\vec{F})\,dV - \iint_{S'} \vec{F}\cdot\hat{n}\,dS$ (↑ divergence thm; R ↳ half ball enclosed by S and S')

$\frac{\partial}{\partial x}(4x^3+x-\sin(yz))$, $\frac{\partial}{\partial y}(-12x^2y+e^{z^3})$, $\frac{\partial}{\partial z}(x^2+y^2)$

$= \iiint_{\text{half ball}} 12x^2 + 1 - 12x^2 + 0 \; dV - \pm\int_0^{2\pi}\int_0^2 \vec{F}\cdot(\vec{r}_r \times \vec{r}_\theta)\,dr\,d\theta$

$= \iiint_{\text{half ball}} 1\,dV - \pm\int_0^{2\pi}\int_0^2 \vec{F}\cdot(\langle\cos\theta, \sin\theta, 0\rangle \times \langle -r\sin\theta, r\cos\theta, 0\rangle)\,dr\,d\theta$

$=$ volume of half-ball of radius 2 $- \ominus\int_0^{2\pi}\int_0^2 \vec{F}\cdot(0,0,r)\,dr\,d\theta$

Choose − since $(0,0,r)$ points up since $r \ge 0$, but $\hat{n}$ is down since $S+S'$ is outwards in divergence thm

$= \frac{1}{2}\cdot\frac{4\pi\cdot 2^3}{3} + \int_0^{2\pi}\int_0^2 (x^2+y^2)\cdot r\,dr\,d\theta$

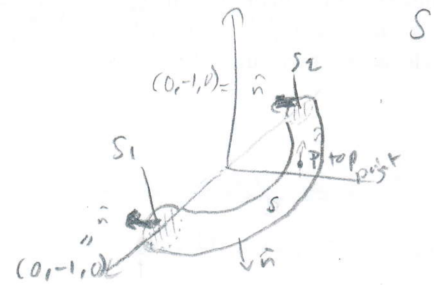
on S', this is equal to r^2 (just plug in parametrization)

$= \frac{16\pi}{3} + \int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta$

$= \frac{16\pi}{3} + 2\pi\left[\frac{r^4}{4}\right]_0^2 = \frac{16\pi}{3} + 8\pi = \frac{40\pi}{3}$

Note: for S', I could have notice $\hat{n} = \langle 0,0,-1\rangle$ since flat surface, and $dS = dx\,dy$. So $\iint_{S'} \vec{F}\cdot\hat{n}\,dS = \iint_{S'} -(x^2+y^2)\,dx\,dy = \int_0^{2\pi}\int_0^2 -r^2\cdot r\,dr\,d\theta = -8\pi$

2. (10 points) Find the volume of the (open at both ends) half-donut parametrized by $\mathbf{r}(u,v) = \langle (3+\cos(v))\cos(u), (3+\cos$ for $u \in [0,\pi]$ and $v \in [0, 2\pi]$. (Hint: use the divergence theorem and an appropriate vector field.)



(not great perspective: S_1 and S_2 are on xz-plane, that is why $\hat{n} = -\hat{j}$)

$S: \vec{r}(u,v) = \langle (3+\cos(v))\cos u, (3+\cos(v))\sin u, \sin(v) \rangle$ $\quad 0 \le u \le \pi$, $0 \le v \le 2\pi$

Want to find volume: $\iiint_D 1\, dV$ (D = half-donut)

But I don't know the function bounding my half-donut, only parametric equations. So use the divergence theorem to make use of parametrization.

div thm if $\text{div}(\vec{F}) = 1$. Find such an $\vec{F}$. For example $\boxed{\langle 0,0,z \rangle}$ or $\langle 0,y,0 \rangle$ or $\langle x,0,0 \rangle$

$$\iiint_D 1\, dV \overset{\text{div thm}}{=} \iint_{S+S_1+S_2} \vec{F}\cdot\hat{n}\, dS \quad \text{where } \hat{n} \text{ is outwards}$$

$$= \pm\iint_S \underbrace{\langle 0,0,\sin v\rangle}_{\vec{F}} \cdot \big(\underbrace{\langle -(3+\cos v)\sin u, (3+\cos v)\cos u, 0\rangle}_{\vec{r}_u} \times \underbrace{\langle -\sin v\cos u, -\sin v \sin u, \cos v\rangle}_{\vec{r}_v}\big)\, du\, dv$$

$$+ \iint_{S_1} \langle 0,0,z\rangle \cdot \langle 0,-1,0\rangle\, dS \quad (= 0)$$

$$+ \iint_{S_2} \langle 0,0,z\rangle \cdot \langle 0,-1,0\rangle\, dS \quad (= 0)$$

At $u = \frac{\pi}{2}$, $v = \frac{\pi}{2}$, $r(u,v) = \langle 0,3,1\rangle = P$ on picture, where $\hat{n} = \langle 0,0,1\rangle$ and $\vec{r}_u \times \vec{r}_v = \langle \sim, \sim, \underbrace{(3+\cos(\frac{\pi}{2}))\sin(\frac{\pi}{2})}_{3} \rangle$, so correct direction

$$= \oplus \int_0^{2\pi}\int_0^{\pi} \langle 0,0,\sin v\rangle \cdot \langle \text{unimportant}, \text{unimportant}, (3+\cos v)\sin v\rangle\, du\, dv$$

$$= \int_0^{2\pi}\int_0^{\pi} (3+\cos v)\sin^2 v\, du\, dv$$

$$= \int_0^{2\pi}\int_0^{\pi} 3\sin^2 v + \sin^2 v \cos v\, du\, dv$$

$\sin^2 v = \frac{1-\cos 2v}{2}$

$$= \pi \int_0^{2\pi} \frac{3}{2}(1-\cos 2v)\, dv + \left[\frac{\sin^3 v}{3}\right]_0^{2\pi} \quad (= 0)$$

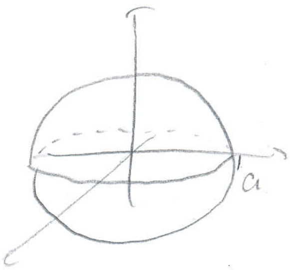
$$= \pi\left(\frac{3}{2}\cdot 2\pi + \frac{3}{2}\cdot\frac{1}{2}\left[\sin 2v\right]_0^{2\pi}\right) \quad (= 0)$$

$$= \boxed{3\pi^2}$$

Could have also realized that when unfolded, this half donut is cylinder

so volume is $\pi \cdot 1^2 \cdot \frac{1}{2}\cdot 2\pi \cdot 3 = 3\pi^2$

3. (10 points) Let S be a sphere of radius a centered at the origin. Calculate $\int\int_S x^2 - z\,dS$ by applying the divergence theorem.



Again, no $\vec{F}$ is given so we need to come up with one that makes the theorem possible.

$$\iint_S x^2 - z\,dS = \iiint_D \text{div}\,\vec{F}\,dV$$

↑ div theorem

if $\vec{F}\cdot\hat{n} = x^2 - z$. outwards for div thm

Since $\hat{n} = \oplus\langle \frac{x}{a}, \frac{y}{a}, \frac{z}{a} \rangle$

$\vec{F}$ could be $\langle ax, 0, -a \rangle$

Then $\text{div}\,\vec{F} = a$

so $\iiint_D \text{div}\,\vec{F}\,dV = \iiint_D a\,dV$

$= a \cdot \iiint_D dV$

$= a \cdot$ volume of D

$= a \cdot \frac{4\pi a^3}{3} = \frac{4\pi a^4}{3}$