On congruences of lines of order 1 in $\mathbb{P}^3$*

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Introduction

In the present paper we reprove a classical result of Kummer, i.e. the classification of order 1 congruences of lines in $\mathbb{P}^3$. Ziv Ran [1] was the first to give a modern proof (and a precise statement) of this result; we hope that our proofs and statements are more geometrically clear than those from [1]; at least our proofs are entirely different.

1 Notation and conventions

The ground field will always be algebraically closed and of characteristic zero.

The Grassmannian of lines in $\mathbb{P}^3$ will be denoted by $\mathbf{G} = G(1,3)$. In the sequel we will identify $\mathbf{G}$ with Plücker quadric in $\mathbb{P}^5$. If $\alpha \in \mathbf{G}$, then $l_\alpha \subset \mathbb{P}^3$ denotes the corresponding line in $\mathbb{P}^3$; if $l \subset \mathbb{P}^3$ is a line, then the corresponding point of $\mathbf{G}$ is denoted by $\alpha_l \in \mathbf{G}$.

If $a \in \mathbb{P}^3$ is a point and $S \subset \mathbb{P}^3$ is a plane, then $\mathbb{P}^2(a) = \{\, \alpha \in \mathbf{G} : l_\alpha \ni a \,\}$ and $\mathbb{P}^2(S) = \{\, \alpha \in \mathbf{G} : l_\alpha \subset S \,\}$. It is clear that $\mathbb{P}^2(a)$ and $\mathbb{P}^2(S)$ are planes in $\mathbf{G}$ and that any plane lying on $\mathbf{G}$ is either $\mathbb{P}^2(a)$ or $\mathbb{P}^2(S)$ for some α or S. The quadric $\mathbf{G}$ does not contain linear spaces of dimension greater than 2. Any line lying on $\mathbf{G}$ is of the form $\mathbb{P}^1(a,S) = \mathbb{P}^2(a) \cap \mathbb{P}^2(S)$, where $a \in S$.

For any subset $\Sigma \subset \mathbf{G}$ put $\tilde{\Sigma} = \bigcup_{\alpha\in\Sigma} l_\alpha \subset \mathbb{P}^3$; if $a \in \mathbb{P}^3$, then put $\Sigma_a = \Sigma \cap \mathbb{P}^2(a)$.

The set of lines meeting a given line $l \subset \mathbb{P}^3$ will be denoted by H_l. It is well known that $H_l = T_{\alpha_l} \cap \mathbf{G}$, where T stands for embedded Zariski tangent space. Such linear sections of the Grassmannian will be called *special linear complexes*.

Definition 1.1. Any irreducible surface $\Sigma \subset \mathbf{G}$ will be called a *congruence of lines* in $\mathbb{P}^3$.

Definition 1.2. If Σ is a congruence of lines in $\mathbb{P}^3$, then its *order* is the number of lines from the congruence passing through a generic point of $\mathbb{P}^3$,

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and its *class* is the number of lines from the congruence lying in a generic plane in $\mathbb{P}^3$.

Since the classes of $[\mathbb{P}^2(a)]$ and $[\mathbb{P}^2(S)]$ generate the Chow group of 2-cycles in $\mathbf{G}$ and $[\mathbb{P}^2(a)] + [\mathbb{P}^2(S)]$ is the class of the codimension 2 linear section of $\mathbf{G}$, we see that $\deg \Sigma = \mathrm{ord}(\Sigma) + \mathrm{cl}(\Sigma)$ for any congruence $\Sigma \subset \mathbf{G}$.

If $\Sigma \subset \mathbf{G}$ is a congruence, then its *dual* congruence is, by definition, the congruence Σ^* in the dual projective space $(\mathbb{P}^3)^*$ defined by the formula $\Sigma^* = \{\, l_\alpha^\perp : \alpha \in \Sigma\,\}$. Order of Σ^* equals class of Σ, and class of Σ^* equals order of Σ.

It is very easy to describe congruences of order or class 0:

Lemma 1.3. *Suppose that $\Sigma \subset \mathbf{G}$ is a congruence in $\mathbb{P}^3$. Then:*

- $\mathrm{cl}(\Sigma) = 0 \Leftrightarrow \Sigma = \mathbb{P}^2(L)$ *for a plane* $L \subset \mathbb{P}^3$*;*
- $\mathrm{ord}(\Sigma) = 0 \Leftrightarrow \Sigma = \mathbb{P}^2(x)$ *for a point* $x \in \mathbb{P}^3$*.*

Proof. If $\mathrm{cl}(\Sigma) = 0$, then the lines of the congruence Σ sweep out a surface in $\mathbb{P}^3$. Since any surface S containing a 2-dimensional family of lines must be a plane (proof: if $x \in S$ is a generic smooth point, then a 1-dimensional family of lines lying on S passes through x; thus $T_xS = S$), we see that $\Sigma = \mathbb{P}^2(S)$. This proves the first statement; to prove the second statement, dualize. $\square$

2 Congruences associated with curves

Let C_1 and C_2 be two (not necessarily distinct) curves in $\mathbb{P}^3$. Then the closure of the set of lines joining generic points of C_1 with those of C_2 is a congruence in $\mathbb{P}^3$. This congruence will be denoted $\Sigma(C_1, C_2) \subset \mathbf{G}$ (or simply $\Sigma(C)$ if $C_1 = C_2$).

It is easy to compute the class of such congruences:

Proposition 2.1. *If $C_1 \neq C_2$ are two curves in $\mathbb{P}^3$ and $\deg C_i = d_i$, then* $\mathrm{cl}(\Sigma(C_1, C_2)) = d_1 d_2$*; if $C \subset \mathbb{P}^3$ is a curve of degree d, then* $\mathrm{cl}(\Sigma(C)) = d(d-1)/2$.

Proof. Let $L \subset \mathbb{P}^3$ be a generic plane. Then $\#(L \cap C_i) = d_i$. Hence, $\#(\Sigma \cap \mathbb{P}^2(L)) = d_1 \cdot d_2$, since any line corresponding to a point of $\Sigma \cap \mathbb{P}^2(L)$ must pass from one point of C_1 and one point of C_2. The computation of $\mathrm{cl}(\Sigma(C))$ is analogous. $\square$

To compute order of $\Sigma(C_1, C_2)$ we need the following definition:

Definition 2.2. Suppose that $C_1 \neq C_2$ are two curves in $\mathbb{P}^3$ and that $x \in C_1 \cap C_2$. If π is a generic projection from $\mathbb{P}^3$ to $\mathbb{P}^2$, then the multiplicity of intersection of $\pi(C_1)$ and $\pi(C_2)$ at the point $\pi(x)$ will be called *multiplicity* of intersection of C_1 and C_2 at x.

If C_1 and C_2 are smooth at x and $T_xC_1 \neq T_xC_2$, then multiplicity of intersection of C_1 and C_2 at x equals 1.

Proposition 2.3. *If $C_1 \neq C_2$ are two curves in $\mathbb{P}^3$ and $\deg C_i = d_i$, then $\operatorname{ord}(\Sigma(C_1, C_2)) = d_1 \cdot d_2 - \#(C_1 \cap C_2)$, where the number of intersection points is computed taking into account multiplicities.*

Proof. Let $x \in \mathbb{P}^3$ be a generic point, and denote the projection with center x by $\pi_x \colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$. If a line of the congruence passes through x, this line yields a point $y \in \tilde{C}_1 \cap \tilde{C}_2$, where $\tilde{C}_i = \pi_x(C_i)$. Since $y \notin \pi(C_1 \cap C_2)$ and $\deg \tilde{C}_i = d_i$, everything follows now from Bezout theorem. □

Now let us prove that the curves C_i can (with trivial exceptions) be recovered from the congruence $\Sigma(C_1, C_2)$.

Proposition 2.4. *Suppose that Σ is a congruence and that $\Sigma = \Sigma(C_1, C_2) = \Sigma(C_1', C_2')$, where C_1, C_2, C_1', C_2' are (not necessarily distinct) curves in $\mathbb{P}^3$. Then either $\Sigma = \mathbb{P}^2(S)$ for some $S \subset \mathbb{P}^3$, or $\{C_1, C_2\} = \{C_1', C_2'\}$.*

Proof. We start with two lemmas.

> **Lemma 2.5.** *Suppose that $B_1 \neq B_2 \subset \mathbf{G}$ are two 1-dimensional irreducible families of lines in $\mathbb{P}^3$ such that for generic $\alpha_1 \in B_1, \alpha_2 \in B_2$ we have $l_{\alpha_1} \cap l_{\alpha_2} \neq \varnothing$. Then either all the lines of both families pass through some point $p \in \mathbb{P}^3$, or both the families of lines sweep the same surface in $\mathbb{P}^3$.*
> *Suppose that $B \subset \mathbf{G}$ is a 1-dimensional irreducible family of lines in $\mathbb{P}^3$ such that for any two generic $\alpha_1, \alpha_2 \in B_1$ we have $l_{\alpha_1} \cap l_{\alpha_2} \neq \varnothing$. Then either $B \subset \mathbb{P}^2(p)$ for some point $p \in \mathbb{P}^3$, or $B \subset \mathbb{P}^2(S)$ for some plane $S \subset \mathbb{P}^3$.*

1. *Proof of Lemma 2.5.* (1) Denote the surface swept by the lines of the family B_i by S_i, and pick a generic $\alpha \in B_1$. If the intersection points vary as β varies in B, then $l_\alpha \cap l_\beta$ $l_\alpha \subset S_2$ and $S_1 = S_2$. Otherwise there exists a point $p \in l_\alpha$ such that all the lines l_β with $\beta \in B_2$ pass through p. Hence, S_2 is a cone with the vertex p. Now it is clear that either $S_1 = S_2$ or all the lines of both families pass through p.

   (2) If $\alpha_1, \alpha_2 \in \mathbf{G}$, then $l_{\alpha_1} \cap l_{\alpha_2} \neq \varnothing$ iff all the line $\overline{\alpha_1, \alpha_2}$ lies in $\mathbf{G}$. Thus, the linear span of B is contained in $\mathbf{G}$. Now everything follows from the description of linear spaces lying on $\mathbf{G}$. □

Lemma 2.6. *Suppose that the curves $C_1, C_2 \subset \mathbb{P}^3$ (we do not suppose that $C_1 \neq C_2$) are such that $T_xC_1 \cap T_yC_2 \neq \varnothing$ for generic $x \in C_1, y \in C_2$. Then there exists a plane $L \subset \mathbb{P}^3$ such that $C_1 \cup C_2 \subset L$.*

Proof of Lemma 2.6. Let D_i stand for a family of lines in $(\mathbb{P}^3)^*$ that is the closure of the family of lines of the form $(T_xC_1)^\perp$, where x passes through the smooth points of C_i. Observe that the lines of D_i sweep out the dual variety $(C_i)^* \subset (\mathbb{P}^3)^*$. The hypothesis implies that a generic line from the family D_1 meets a generic line of the family D_2. Lemma 2.5 implies that either both $(C_1)^*$ and $(C_2)^*$ are cones with the same vertex, or $(C_1)^* = (C_2)^*$. In the former case, projective duality implies that C_1 and C_2 lie in the same plane; in the latter case Lemma 2.5 implies that $(C_1)^*$ lies in a plane, whence $C_1 = C_2$ is a line. □

Return now to the proof of the proposition. Assuming that $\Sigma \neq \mathbb{P}^2(S)$ for a plane S, it suffices to show that $\{C_1', C_2'\} \subset \{C_1, C_2\}$. Assume the contrary; one may assume that the curve C_1' is distinct from both C_1 and C_2. For generic $x \in C_1$ and $y \in C_2$ we see that $\overline{xy} = \overline{zw}$, where $z \in C_1'$ and $w \in C_2'$. Since $C_1' \notin \{C_1, C_2\}$, we see that the point z varies as x varies in C_1 or y varies in C_2. Hence, for generic $x \in C_1$ and $y \in C_2$ we have $C_1' \subset x * C_2$ and $C_1' \subset y * C_1$. Thus,

$$T_wC_1' \subset (x * T_yC_1) \cap (y * T_xC_2). \tag{2.1}$$

If the linear space in the right-hand side of (2.1) is two-dimensional, then $T_xC_1 \cap T_yC_2 \neq \varnothing$, and Lemma 2.6 implies that C_1 and C_2 lie in the same plane S, whence $\Sigma = \mathbb{P}^2(S)$. If the linear space in question is one-dimensional, then it coincides with the line $\overline{xy}$ which is tangent to the curve C_1'. Thus, all the lines of our congruence are tangent to the curve $C'1$, which is clearly impossible. This contradiction completes the proof. □

3 Examples of congruences of order 1

In this section we give some examples of congruences of order 1 in $\mathbb{P}^3$. Instead of computation in coordinates, we will try to give geometric argument.

Example 3.1. Let l_1 and l_2 be two skew lines. Then it is clear that $\Sigma(l_1, l_2)$ is a congruence of order and class 1. Let us show that the surface $\Sigma(l_1, l_2) \subset \mathbf{G}$ is a smooth quadric which is a codimension 2 linear section of $\mathbf{G}$. Indeed, since $\Sigma(l_1, l_2) = H_{l_1} \cap H_{l_2}$, we see that $\Sigma(l_1, l_2)$ lies in a linear space of codimension two and hence is an irreducible quadric; if this quadric is a cone, then there is only one line of the congruence (the line corresponding to the vertex) meeting all the other lines, which is not the case. Hence, our quadric is smooth.

Example 3.2. Put $\Sigma = \Sigma(l, K)$, where k is a conic, l is a line not lying in the plane of the conic K, and $K \cap l = z$. It follows from Propositions 2.1 and 2.3 that our congruence has order 1 and class 2. Let us show that $\Sigma \subset \mathbf{G} \subset \mathbb{P}^5$ is projectively isomorphic to the smooth ruled surface of degree 3 in $\mathbb{P}^4$.

To that end, pick a point $y \in l \setminus z$ and consider the curves $K_1, K_2 \subset \mathbf{G}$ defined as follows: $\alpha \in K_1$ iff there exists a point $x \in K$ such that $l_\alpha = \overline{xy}$, and $K_2 = \mathbb{P}^2(z, \langle K \rangle)$. It is clear that K_1 is a conic and K_2 is a line, and that $\langle K_1 \rangle \cap K_2 = \varnothing$. The mapping $\varphi\colon K_1 \to K_2$ that assigns the line $\overline{zx}$ to a line $\overline{yx}$, is an isomorphism from K_1 to K_2, and

$$\Sigma = \bigcup_{\alpha \in K_1} \overline{\alpha, \varphi(\alpha)} \subset \mathbf{G}. \tag{3.1}$$

The equation (3.1) is a definition of the smooth ruled surface of degree 3 in $\mathbb{P}^4$.

Example 3.3. Generalizing the above example, let $C \subset \mathbb{P}^3$ be a curve of degree n and let $L \subset \mathbb{P}^3$ be such a line that the sum of multiplicities (in the sense of Definition 2.2) of intersection points of L and C equals $n-1$. It is easy to see that the curve C must be rational and smooth (project from L). Then Propositions 2.1 and 2.3 imply that $\Sigma(C, L)$ is a congruence of order 1 and class n.

Remark 3.4. The following construction of curves meeting the requirements of Example 3.3 was worked out in a discussion with Carlos Simpson. Let $C' \subset \mathbb{P}^2$ be a curve of degree n with a point $p \in C'$ of multiplicity $n-1$ (in other words, let C" be the projective closure of the affine curve with the equation $F(x,y) + G(x,y) = 0$, where F and G are binary forms of degrees n and $n-1$ respectively). Let $\overline{\mathbb{P}^2}$ be the blow-up of $\mathbb{P}^2$ at p and let $\overline{C} \subset \overline{\mathbb{P}^2}$ and $\overline{L} \subset \overline{\mathbb{P}^2}$ be the proper inverse image of C in $\overline{\mathbb{P}^2}$ and the exceptional curve, respectively. Embed $\overline{\mathbb{P}^2}$ in $\mathbb{P}^4$ as a ruled surface of degree 3 and project from a generic point $x \in \overline{C}$. The curve $C = \pi_x(\overline{C})$ and the line $L = \pi_x(\overline{L})$ satisfy the hypothesis of Example 3.3.

Example 3.5. Let $l \subset \mathbb{P}^3$ be a line, and let $\varphi\colon l^\perp \to l$ be an arbitrary non-constant morphism. Put $\Sigma(l, \varphi) = \bigcup_{\beta \in l^\perp} \mathbb{P}^1(\varphi(\beta), \mathbb{P}^2_\beta)$. It is easy to see that $\Sigma(l, \varphi)$ is a congruence of order 1 and class $\deg \varphi$.

Example 3.6. Let $C \subset \mathbb{P}^3$ be the twisted cubic. Let us show that $\Sigma = \Sigma()$ is a congruence of order 1 and class 3.

Indeed, in view of Proposition 2.1 it suffices to compute the order. To see that the order equals one, observe that $l_1 \cap l_2 = \varnothing$ whenever $l_1 \neq l_2$ and both of the l_i's are secants or tangents of the curve C. Indeed, if $l_1 \cap l_2 \neq \varnothing$, then the intersection index of C with the plane spanned by l_1 and l_2 is at least 4, which is impossible.

Let us show that the surface $\Sigma \subset \mathbf{G} \subset \mathbb{P}^5$ is projectively isomorphic to the Veronese surface in $\mathbb{P}^5$.

To begin with, observe that $\deg \Sigma = \operatorname{ord} \Sigma + \operatorname{cl} \Sigma = 4$.

Let us show now that Σ is the image of $\mathbb{P}^2$ under a morphism. Indeed, The linear system of quadrics containing C is two-dimensional. Denote this linear system by $L \cong \mathbb{P}^2$. Define the map $\varphi\colon L^* \to \mathbf{G}$ as follows: any $\beta \in L^*$

is a pencil of quadric containing C; the base locus of this pencil is a union of C and a line l_α, where $\alpha \in G$. Put $\varphi(\beta) = \alpha$. It is clear that the l_α's are either secants or tangents of the curve C, so $\varphi(L^*) \subset \Sigma$. Since the cone $x * C$ is a quadratic cone for any point $x \in C$, we see that, for any two distinct points $x, y \in C$, the line $\overline{xy}$ is contains in the intersection of quadrics $x * C$ and $y * C$. Thus, the mapping $\varphi\colon L^* \to \Sigma$ is surjective.

Let us show now that $\Sigma \subset \mathbb{P}^5$ is not contained in a hyperplane. Indeed, assume that $\Sigma \subset H$, where H is a hyperplane in $\mathbb{P}^5$. Pick a generic line $l_1 \subset \mathbb{P}^3$; then $H_{l_1} \cap H \cap \mathbf{G} = H_{l_1} \cap H_{l_2} \cap \mathbf{G}$ for a line $l_2 \subset \mathbb{P}^3$. Pick another generic line $l_3 \subset \mathbb{P}^3$. Then $H_{l_1} \cap H_{l_2} \cap H_{l_3} \cap \Sigma$ is a generic linear section of codimension 2 of the surface Σ; since $\deg \Sigma = 4$, this set consists of four distinct lines, each of which is a tangent or a secant to C. On the other hand, let Q be a smooth quadric in $\mathbb{P}^3$ which is swept by lines meeting l_1, l_2, and l_3; let us show that $C \subset Q$. Indeed, if this is not the case, then the intersection index of C and Q is at least $4 \cdot 2 = 8$, which is impossible. Since $C \subset Q$ is a twisted cubic and $l_1 \subset Q$, we see that $l_1 \cap C \neq \varnothing$; this yields the desired contradiction (the generic l_1 cannot meet C).

Thus, $\Sigma \subset \mathbb{P}^5$ is a surface of degree 4, Σ does not lie in a hyperplane and is an image of $\mathbb{P}^2$ under a morphism. Hence, Σ is the Veronese surface.

Observe that, among all the congruences described in this section, only $\Sigma(C)$ with C a twisted cubic does not correspond to a surface in $\mathbb{P}^5$ not lying in a hyperplane.

4 Classification of order 1 congruences

Let us show that there exist no congruences of order 1 in $\mathbb{P}^3$ except those described in the previous section.

We will start with the following definition:

Definition 4.1. Let $\Sigma \subset \mathbf{G}$ be a congruence in $\mathbb{P}^3$. A point $x \in \mathbb{P}^3$ is called a *jump point* if $\dim \mathbb{P}^2(x) \cap \Sigma > 0$. The set of all jump points is called *jump locus* of the congruence.

Remark 4.2. If Σ has order 1 and if more than one line of the congruence passes through a point $p \in \mathbb{P}^3$, then Zariski main theorem implies that the point p lies in the jump locus of Σ.

Proposition 4.3. *If Σ is a congruence of order 1, then its jump locus is not empty.*

Proof. Pick a point $\alpha \in \Sigma$ and choose a curve $K \subset \Sigma$ such that $\alpha \notin C$. Then the non-empty set $(\bigcup_{\beta \in C} l_\beta) \cap l_\alpha$ is contained in the jump locus. $\square$

Now we describe irreducible components of the jump locus of an order 1 congruence.

Lemma 4.4. *Suppose that Σ is a congruence of order 1 and that Z is its jump locus. Then:*

1. *If $\dim Z = 0$, then $\Sigma = \mathbb{P}^2(x)$ for some point $x \in \mathbb{P}^3$*
2. *If $\dim Z = 1$, then either $Z = C$ or $Z = C_1 \cup C_2$, where $C, C_1, C_2 \subset \mathbb{P}^3$ are irreducible curves.*
3. *If $Z = C_1 \cup C_2$, where C_i are irreducible curves, then $\Sigma = \Sigma(C_1, C_2)$.*
4. *If $Z = C$ is an irreducible curve and not a line, then $\Sigma = \Sigma(C)$.*
5. *If $Z = l$ is a line, then $\Sigma = \bigcup_{\beta \in l^\perp} \mathbb{P}^1(\varphi(L), L_\beta)$, where $l^\perp$ is the set of planes containing the line l and $\varphi\colon l^\perp \to l$ is a non-constant morphism.*

Proof. For any point $p \in \mathbb{P}^2$, put $S(p) = \bigcup_{\alpha \in \Sigma, l_\alpha \ni p} l_\alpha$.

It is clear that the variety Z cannot contain two-dimensional components. Let us show that C cannot contain both zero-dimensional and one-dimensional components. Indeed, assuming the contrary, suppose that a curve K and a point a are both components of Z. Since σ is irreducible, any line of the congruence meets K. Let b and c be two different generic points of the curve K. Since $S(a)$ is one-dimensional, the lines $\overline{ab}$ and $\overline{ac}$ belong to the congruence. Hence the set $S(b) \cap S(c)$ is one-dimensional, is contained in Z and contains a, which is impossible since a is a component of Z.

If $Z(\Sigma)$ consists of a finite number of points, then there is at least one of them (say a) through which a two-dimensional family of lines of the congruence passes. Hence, $\Sigma = \mathbb{P}^2(a)$ and $Z = \{a\}$, which proves statement (1).

Suppose now that $\dim Z = 1$. Then all the irreducible components $K_i \subset Z$ are curves, and for any i the equality $\Sigma = \bigcup\limits_{x \in K_i} (\mathbb{P}^2(x) \cap \Sigma)$ holds.

Hence, if $i > 1$, then $\Sigma = \Sigma(K_{i_1}, K_{i_2})$ for any $i_1 \neq i_2$; Proposition 2.4 shows that Z consists of exactly two components K_1 and K_2 and that $\Sigma = \Sigma(K_1, K_2)$. Thus, statements (2) and (3) are proved.

It suffices to consider the case when Z is an irreducible curve. Suppose that $x_1, x_2 \in Z$ and $x_1 \neq x_2$. Let us show that $\alpha_{\overline{x_1,x_2}} \in \Sigma$. Indeed, $S(x_1) \cap S(x_2)$ is a curve contained in Z. Since Z is irreducible, this curve coincides with Z, so $x_2 \in S(x_1)$ and $\alpha_{\overline{x_1,x_2}} \in \Sigma$.

If Z is not a line, then it follows immediately from what we have just proved that $\Sigma = \Sigma(Z)$. This yields statement (4).

Assume at last that Z is a curve; let $L \supset Z$ be a plane. Then all the lines of the congruence that are contained in L, can intersect only in the points of Z; hence, all such lines pass through one point $a_L \in Z$. Thus, to any plane containing Z we assigned a unique point of Z. Hence, $\Sigma = \Sigma(Z, \varphi)$, where $\varphi\colon Z^\perp \to Z$ is the morphism defined by the rule $\beta \mapsto a_{L_\beta}$. This finishes the proof of statement (5) and the lemma. $\square$

Lemma 4.5. *If* $C \subset \mathbb{P}^3$ *is a curve such that* $\operatorname{ord}(\Sigma(C)) = 1$*, then* $\deg(C) = 3$.

Proof. Assuming the contrary, suppose that $\deg C > 3$. Then, by virtue of Harris' uniform position lemma, a generic plane section of the curve C contains at least four points, no three of which are collinear. Thus, this section contains four points a, b, c, d such that the point $p = \overline{ac} \cap \overline{bd}$ does not lie on C. It is clear that $p \in Z$; Lemma 4.4 implies that $\Sigma(C) = \Sigma(C, C')$, where $C' \neq C$. This contradicts the hypothesis. $\square$

Lemma 4.6. *If the curves* $C_1, C_2 \subset \mathbb{P}^3$ *are such that* $\operatorname{ord}(\Sigma(C_1, C_2)) = 1$*, then one of the* C_i*'s is a line.*

Proof. Assuming the contrary, suppose that $\deg C_i > 1$ for $i = 1, 2$. Let $L \subset \mathbb{P}^2$ be a generic plane; put $L \cap C_i = M_i$. It is clear that $M_1 \cap M_2 = \varnothing$, and there exist points $a, b \in M_1$ and $c, d \in M_2$ such that the point $p = \overline{ac} \cap \overline{bd}$ does not lie in $M_1 \cap M_2$. Hence, the point p belongs to the jump locus of the congruence $\Sigma(C_1, C_2)$; this contradicts Lemma 4.4. $\square$

Summing up, we obtain the main result:

Theorem 4.7. *Any congruence of order* 1 *in* $\mathbb{P}^3$ *is either the set of lines passing through a given point, or the set of all the secants and tangents of a twisted cubic, or is obtained by one of the constructions described in examples 3.3 and 3.5.*

References

[1] Z. Ran. Surfaces of order 1 in Grassmannians. *J. Reine Angew. Math.*, 368:119–126, 1986.