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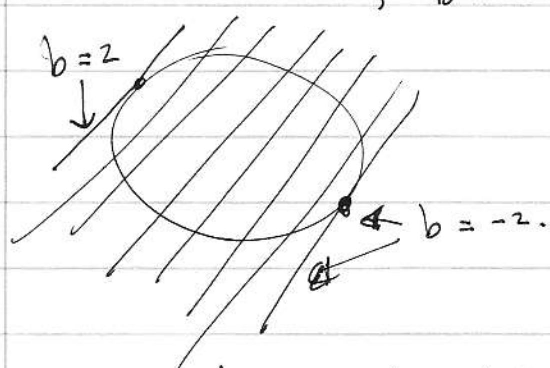
3.4.3. $x = t$, $y = b + t$ $\longrightarrow$ $(x^2 + y^2 = 2)$ $\quad t^2 + (b+t)^2 = 2$

$$2t^2 + b^2 + 2bt = 2$$

$$2t^2 + 2bt + (b^2 - 2) = 0 \quad *$$

This equation (*) has 2 roots except when $(2b)^2 - 4(b^2-2)2 = 0$ (using the quadratic formula) or ~~$4b^2 - 8b^2 + 16 = 0$~~ $4b^2 - 8b^2 + 16 = 0$

$16 = 4b^2$, $b = \pm 2$.



3.4.5. (a) t is a root of multiplicity $\geq 2 \Rightarrow g'(0) = 0$.

proof. t is a root of $g(t)$ of mult ≥ 2 $\Rightarrow$ t^2 divides $g(t)$ $\Leftrightarrow$ $g(t) = t^2 h(t)$ for some polynomial $h(t)$ $\Rightarrow$ $g'(t) = t^2 h'(t) + h'(t) \cdot 2t$. Evaluate at $t = 0$ and one gets $g'(0) = 0$

$g'(0) = 0 \Rightarrow t$ is a root of multiplicity ≥ 2.

proof. $t = 0$ is a root $\Rightarrow$ t divides $g(t)$ $\Rightarrow$ $g(t) = t \cdot h(t)$. $g'(t) = t h'(t) + h(t)$

$g'(0) = 0 \cdot h'(0) + h(0) = 0$ (hypothesis) $\Rightarrow h(0) = 0$

$\Rightarrow h(t) = t\, k(t)$ some polynomial $k(t)$

$\Rightarrow g(t) = t \cdot h(t) = t \cdot t\, k(t) = t^2 k(t) \Rightarrow t$ is a root of mult ≥ 2.

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4.3.5b. $\left(\begin{array}{c} t \text{ is a root of mult } k \\ \text{of } g(t) \end{array} \right) \Leftrightarrow \left[g(0) = g'(0) \cdots = g^{k-1}(0) = 0 \right]$

By induction, ~~the case~~ The case $k=2$ has been done in 4.3.5a.
~~We know~~ We have $\left[\begin{array}{c} t \text{ is a root of mult } (k-1) \text{ of} \\ h(t) \end{array} \right] \Leftrightarrow \left(h(0) = \cdots = h^{(k-2)}(0) = 0 \right)$

by inductive hypothesis.

Lemma: $t=0$ is a root of mult k of $g(t) \Leftrightarrow g(0) = 0$ and t is a root of mult $(k-1)$ of $g'(t)$.

Proof of lemma:
$\Rightarrow$: $t=0$ root of mult $k \Leftrightarrow g(t) = t^k h(t)$ (by definition of multiplicity) $\Rightarrow g'(t) = t^k h'(t) + k t^{k-1} h(t) \Rightarrow t^{k-1}$ divides $g'(t) \Rightarrow t$ is a root of mult $(k-1)$ of $g'(t)$.

$\Leftarrow$: t is a root of mult $k-1$ of $g'(t) \Rightarrow g'(t) = t^{k-1} \cdot h(t)$. now integrate, get ~~g(t)~~ $g(t)$ is divisible by t^k.

Proof of 4.3.5b $t=0$ is a root of mult k of $g(t) \Leftrightarrow t$ is a root of mult $k-1$ of $g'(t) \Leftrightarrow g'(0) = \cdots = g^{k-1}(0) = 0$ by induction. QED

Another proof. We show $g(t)$ has a zero of multiplicity exactly $k \Leftrightarrow$
$$g(0) = \cdots = g^{(k-1)}(0) = 0, \quad g^{(k)} \neq 0.$$
This suffices. (Why?)

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This denotes the k-th derivative

Lemma: u, v = functions of t. Then

$$(uv)^{(k)} = u^{(k)}v + \binom{k}{1}u^{(k-1)}v^{(1)} + \cdots \binom{k}{i}u^{(k-i)}v^{(i)} + \cdots + uv^{(k)}.$$

Proof. By induction.

Proof. 3.4.5b. If $g(t)$ has multiplicity k exactly then $g(t) = t^k h(t)$ has i-th derivative $(i \le k)$ of the form (using lemma):

$$g^{(i)} = k(k-1)\cdots(k-i+1)\,t^{k-i}h + \left(\text{terms with higher powers of } t.\right)$$

~~i(i-1)~~ ...

This means that $g^{(i)}(0) = 0$ because a power of t divides $g^{(i)}$ for $i < k$. If $i = k$ we have

$$g^{(k)} = k!\,h + \text{higher powers of } t, \text{ so } g^{(k)}(0) = k!\,h(0)$$

If g is exactly multiplicity k at $t=0$, then get $g^{(k)}(0) \neq 0$ since $h(0) \neq 0$ in that case.

This shows that g has multiplicity k at $t=0 \Rightarrow g(0) = \cdots = g^{(k-1)} = 0$

On the other hand it shows that if g has mult $< k$ at $t = 0$ then $g^{(i)}(0) \neq 0$ for some $i < k$. Thus $g^{(i)} = 0$ for $i = 0, \ldots k-1$ implies the mult is $\geq k$.

3.4.7 (a), $L: \binom{x}{y} = \binom{a}{f(a)} + t\binom{c}{d}$. What is the intersection of L with $y - f(x) = 0$.

$$(f(a) + td) - f(a+ct) = 0$$ Use Taylor series. We get

$$(*) \quad f(a) + td - \left[f(a) + f'(a)ct + \frac{f''(a)}{2} c^2t^2 + \cdots \right] = 0$$

$$(**) \quad t\left[d - f'(a)\cdot c\right] - t^2 \cdot \frac{f''(a)}{2} \cdot c^2 - t^3 \frac{f'''(a)}{3!} + \cdots = 0.$$

Note. This has multiplicity $\geq 2 \iff d - f'(a)c = 0 \iff f'(a) = d/c$
$\iff$ slope of L is $f'(a)$

(b). The above formula $(**)$ gives this. The coefficient of t^2 is $\frac{f''(a)}{2}$.

(c). Follows from $(**)$ in a similar fashion.

3.4.7 (a). We do the problem using the hint of Cox Little O'Shea:
Let L be the line $\binom{x}{y} = \binom{a}{f(a)} + t\binom{1}{f'(a)}$.

This has slope $f'(a)$ and at $t=0$ passes through $(a, f(a))$. It is the classical tangent line. We intersect with $y - f(x) = 0$ we get

$g(t) = (f(a) + t f'(a)) - f(a+t) = 0$.

The mult of intersection is the number k so that $g(0) = g'(0) = \cdots = g^{(k-1)}(0) =$

But $g(0) = 0$ automatically

$g'(0) = f'(a) - f'(a) = 0$ (So this line is the tangent line. This does part (a))

$g''(0) = -f''(a)$. So multiplicity is $\geq 3 \iff f''(a) = 0$
This is part (b)

$g'''(0) = -f'''(a)$. This leads to part (c).