Math 425

Notes 12: Curl and Multiple Integrals

Lecture March 2

Definition 1. Let

$$F : \mathbb{R}^3 \to \mathbb{R}^3, \ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$$

be a vector field. Then *curl* F is

$$curl\ F = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{pmatrix}.$$

Example 1. We compute the curl of the vector field $V(x, y, z) = (-y^2, x, 0)$.

We now try to explain what this means. By definition, the circulation of a vector field F around a closed curve is gotten by breaking up the curve into small portions P_i to P_{i+1} multiplying the component of F in the direction of the segment P_i to P_{i+1} and multiplying by the length of of this segment. Then one adds over all the little segments. Of course then one takes the limit as the length of the segments goes to zero for all segments. This is how one computes work done by a force field if a particle moves around the curve.

We limit ourselves to a vector field that is entirely in the $x - y$ plane. You can think of this as having a third component equal to zero. We compute the circulation of vector field $F : \mathbb{R}^2 \to \mathbb{R}^2, \ (x, y) \mapsto (F_1, F_2)$ around a very small square whose bottom left point is at (x, y). Let h be the length of a side of the square. Label the sides B=bottom, R= right side, T= top side, L=left side. The contribution of B to the circulation is hF_1, while the contribution of T is

$$h(-F_1 - h\frac{\partial F_1}{\partial y}).$$

Thus the net contribution of the top and bottom is $-\frac{\partial F_1}{\partial y}h^2$. Similarly the contribution of the left and right sides is $\frac{\partial F_2}{\partial x}h^2$. Thus the circulation per unit area is approximately

$$-\frac{\partial F_1}{\partial y} + \frac{\partial F_2}{\partial x}.$$

This is the magnitude of *curl* F in this situation, that is, it is the circulation per unit area.

Let P be a point in $\mathbb{R}^3$. We explain the vector *curl* F at P. For a plane $\mathcal{P}$ containing P let C be a small closed curve contained in plane $\mathcal{P}$ and let n denote a unit normal to the plane. The curl of a vector field F at P is the vector that satisfies

$$< curlF, n >= \text{ circulation of } F \text{ around } C \text{ per unit area inside } C \text{ for the plane } \mathcal{P}.$$

In our 2 dimensional example, we obtain *curl* $F = (0, 0, -\frac{\partial F_1}{\partial y} + \frac{\partial F_2}{\partial x})$.

Notation: $\nabla \cdot F = div(F), \ \nabla \times F = curl\ F$.

We establish one property relating the divergence and curl.

THEOREM 2. $div(curl F) = \nabla \cdot (\nabla \times F) = 0$.

Example of Orthogonal Streamlines

EXAMPLE 3. Consider

$$F : \mathbb{R}^2 \to \mathbb{R}^2, \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}.$$

Observe that $\nabla u = (2x, -2y)$ is orthogonal to $\nabla v = (2y, 2x)$. Since ∇u is orthogonal to ∇v and also to the level curves $u(x, y) = c, c \in \mathbb{R}$ we see that ∇v is parallel to the level curves of u. Thus we conclude that the streamlines for ∇v are the level curves of u. Reciprocally, the level curves of v are streamlines for the vector field ∇u. We can find many examples like this. They can be applied in many situations.

Multiple Integrals: Defintion

Notation: Let $x_i, x_{i-1} \in \mathbb{R}$, then $\Delta x_i = x_i - x_{i-1}$.

Definition 2. Let R be the rectangle $[a, b] \times [c, d] \subseteq \mathbb{R}^2$. Let $f : R \to \mathbb{R}$ be a function on R. Let $x_i = a + \frac{i}{n}(b - a), y_j = c + \frac{j}{n}(d - c)$. Then

$$\int\int_R f dxdy = \lim_{n\to\infty} \sum_{i,j=1}^{i,j=n} f(x_i, y_j)\Delta x_i \Delta y_j$$

provided the limit exists.

THEOREM 4. *Assume that f is continuous on a rectangle R except on points in the union of a finite number of graphs. Then the limit in the definition above exists.*

Assume that $f(x, y) = z$ is the graph of a continuous (except possibly on points in the finite union of graphs) non-negative function. Then $\int\int_R f dxdy$ is the volume of the solid above the $z = 0$ plane and below the graph of f.

Here is a fancier case. How do we find the volume under the function $f(x, y)$, above the plane $z = 0$ and within the triangle with vertices $(0, 0), (1, 0), (1, 1)$. Apparently this is not covered by the above theorems. We extend the function f by making it equal to zero outside the triangle and using the rectangle $[0, 1] \times [0, 1]$. The singularities are all on the line $y = x$ and hence are allowable.

How do we compute these double integrals. These integrals are the limits of sums in a rectangular array. Here are two ways of organizing the summing of numbers in a rectangular array. The first method is to add the numbers in each column and then adding these sums. A second method is to add the numbers in each row and then adding these sums.

Formally we have

$$\begin{aligned} \sum_{i,j=1}^{i,j=n} f(x_i, y_j)\Delta x_i \Delta y_j &= \sum_{j=1}^{j=n} \left(\sum_{i=1}^{i=n} f(x_i, y_j)\Delta x_i \right) \Delta y_j \\ &= \sum_{i=1}^{i=n} \left(\sum_{j=1}^{j=n} f(x_i, y_j)\Delta y_j \right) \Delta x_i. \end{aligned}$$

If we take the limit as $n \to \infty$, then these three sums have the same limit provided f is bounded, and continuous outside a finite union of graphs. This is a delicate result.

THEOREM 5. *Assume that f is nice on a rectangle $R = [a,b] \times [c,d]$. The first sum converges to*

$$\int\int_R f dxdy.$$

The second converges to

$$\int_c^d (\int_a^b f dx) dy.$$

The third converges to

$$\int_a^b (\int_c^d f dx) dy.$$

These are all equal.

EXAMPLE 6. Let $R = [-1,2] \times [3,4],\ f(x,y) = 2x^2 + xy + y^2$. Then

$$\int\int_R = \int_3^4 (\int_{-1}^2 f(x,y)dx)dy.$$

We separate out the inside integral to get

$$\int_{-1}^2 (2x^2 + xy + y^2)dx = x^3 + x^2y + xy^2 \mid_{x=-1}^{x=2} = 3y^2 + (3/2)y + 9.$$

and the whole integral is

$$\int_{y=3}^{y=4} (3y^2 + (3/2)y + 9)dy = LTS.$$

We can also integrate with respect to y first.

EXAMPLE 7. We find the volume of the tetrahedron bounded by the planes

$$x = 0,\ y = 0,\ z = 0,\ y - z + x = 1.$$

The first step is to sketch this. We sketch the intersection of the plane $y - z + x = 1$ with each of the planes $x = 0, y = 0, z = 0$. We find that the tetrahedron lies below the $z = 0$ plane. Indeed, it leis below the triangle T in the $z = 0$ plane with vertices $(0,0), (1,0), (0,1)$.

Up to this point we have performed double integrals only over rectangles. We deal with integrating over a triangle theoretically by extending the integrand by the value 0. We are only able to anti-differentiate functions given by a single formula such as $e^x, x^2 - 7x + 1$. Here is how we get out of this difficulty. Assume that $f(x)$ given by a formula from $x = a$ to $x = b$ and is given by zero from $x = b$ to $x = c$. Then $\int_a^c f dx = \int_a^b f dx$.

Let $f(x, y) = 1-x-y$. The function f gives the height of the tetrahedron. We integrate with respect to y first and then x. We get

$$\int_{x=0}^{x=1} \left(\int_{y=0}^{y=1-x} f dy \right) dx.$$

We could also integrate with respect to x first and then with respect to y. In this case we get

$$\int_{y=0}^{y=1} \left(\int_{x=0}^{x=1-y} f(x) dx \right) dy.$$