Notes 11: Dimension, Rank Nullity theorem

Lecture Oct , 2011

Definition 1. Let V be a subspace of $\mathbb{R}^n$. A basis of V is a subset S of V provided

- the set S spans V, and
- the set S is independent.

Definition 2. Let V be a subspace of $\mathbb{R}^n$. The dimension of V is the number of elements in a basis of V.

Example 1. The set $\{e_1 = (1,0), e_2 = (0,1)\}$ is a basis of $\mathbb{R}^2$. The dimension of $\mathbb{R}^2$ is 2.

Example 2. The set $\{u = (1,2), v = (-2,3)\}$ is a basis of $\mathbb{R}^2$. To show this is the case we have to show two things.

- Every equation of the form
$$x\begin{pmatrix}1\\2\end{pmatrix} + y\begin{pmatrix}-2\\3\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$$
in unknows x, y can be solved. This says that our set spans $\mathbb{R}^2$.
- The only relations of the form
$$au + bv = 0$$
are the ones with $a = b = 0$. This says that the set is independent.

Example 3. Let e_i denote the element of $\mathbb{R}^n$ with all zero entries except in the i-th position where a 1 occurs. Then $S = \{e_1, e_2, \cdots, e_n\}$ is a basis of $\mathbb{R}^n$. The dimension of $\mathbb{R}^n$ is n.

Example 4. We show how to find a basis of the kernel of a matrix. We examine our algorithm for finding the kernel of a matrix. For example if we start with the matrix

$$M = \begin{pmatrix}1 & 1 & 1 & 1 & 1\\ 1 & 0 & -1 & 2 & -2\\ 3 & 2 & 1 & 4 & 0\end{pmatrix}.$$

We find that its RREF is

$$\begin{pmatrix}1 & 0 & -1 & 2 & -2\\ 0 & 1 & 2 & -1 & 3\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}.$$

From this we see that the kernal is all the linear combinations

$$a\begin{pmatrix}1\\-2\\1\\0\\0\end{pmatrix} + b\begin{pmatrix}-2\\1\\0\\1\\0\end{pmatrix} + c\begin{pmatrix}2\\-3\\0\\0\\1\end{pmatrix}.$$

This says that the set

$$S = \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

spans the kernel.

I claim that the set is independent. Take an arbitrary linear combination of the elements as above and do the addition. You get

$$\begin{pmatrix} * \\ * \\ a \\ b \\ c \end{pmatrix}.$$

For this to be zero we must have $a = b = c = 0$. This says that the only linear relation among the three elements is the trivial one. The number of elements in the basis of the kernel is the number of free variables. We conclude

The Dimension of the Kernel of a Matrix is the Number of Free Variables of that Matrix

EXAMPLE 5. The set

$$B = \{v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}\}$$

is not a basis. One reason is that $v_1 + v_2 - v_3 = 0$. This is a non-trivial linear relation. We can find such relations by setting up the matrix

$$M = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

and computing the kernel of M. Upon row-reduction we obtain

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

From this we find the element

$$\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

in the kernel. This gives the relation $-v_1 - v_2 - v_3 = 0$ on the columns of M.

EXAMPLE 6. Let $S = \{v_1, v_2, \cdots, v_t\} \subset \mathbb{R}^n$. Assume that $< v_i, v_j >= 0$, if $i \neq j$, and assume that none of the $v_i = 0$. Then S is independent. We have to show that if we have a linear relation

$$\sum a_i v_i = 0$$

then all the $a_i = 0$. To see this we dot the relation with v_j. Since $< v_i, v_j >= 0$ for all i except $i = j$ we get $a_j < v_j, v_j >=< 0, a_j >= 0$. Since $< v_j, v_j >\neq 0$, this implies that $a_j = 0$.

Facts about Dimension

1. If V is a subspace of $\mathbb{R}^n$, then every basis of V has the same number of elements and this number is finite.

2. The number of elements in a subspace is ∞ except when the subspace is just the single vector 0.

3. Let V, W be subspaces of $\mathbb{R}^n$. If $V \subset W$, then $dimV \leq dimW$. If $V \subset W$ and $dimV = dimW$, then $V = W$.

Let M be an $m \times n$ matrix. We have defined the rank of M to be the number of leading ones in the RREF of M. We examine our algorithm for finding a basis of $im(M)$. We start with the set of m column vectors of M and we remove some of them. Indeed we remove the columns corresponding to the free variables. These are the columns that do not have a leading one. The columns remaining are the ones corresponding to the columns with leading ones. We conclude that the number of elements in our basis of $im(M)$ is the rank. We conclude:

The Rank of a Matrix is the Dimension of the Image

Rank-Nullity Theorem

Since the total number of variables is the sum of the number of leading ones and the number of free variables we conclude:

THEOREM 7. *Let M be an $n \times m$ matrix, so M gives a linear map*

$$M : \mathbb{R}^m \to \mathbb{R}^n.$$

Then

$$m = dim(im(M)) + dim(ker(M)).$$

This is called the rank-nullity theorem. The dimension of the kernel of a matrix is called the nullity. The kernel is called the null space.

Definition 3. let $f : A \to B$ be a function (so the domain of f is the set A and the range or target or codomain is B). We say that f is onto if for every $y \in B$, there is an $x \in A$ so that $f(x) = y$.

This is equivalent to saying that every equation of the form $f(x) = y$ has a solution. If M is an $n \times m$ matrix, then $M : \mathbb{R}^m \to \mathbb{R}^n$ is onto provided the number of leading ones is n.

Definition 4. Let $f : A \to B$ be a function. We say that f is one-to-one or $1 - 1$ if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

This is equivalent to saying that if an equation $f(x) = y$ has a solution, then it has a unique solution. If M is an $n \times m$ matrix, then $M : \mathbb{R}^m \to \mathbb{R}^n$ is one-to-one provided the number of free variables is zero. This is equivalent to saying that the kernel is just the zero vector.

COROLLARY 1. *Let M be an $n \times n$ matrix. Assume that* $\ker(M) = \{0\}$. *The M is onto.*

COROLLARY 2. *Let M be an $n \times m$ matrix. If $m \geq n$ and* $\dim \ker(M) \leq m - n$, *then M is onto.*

COROLLARY 3. *Let Mbe an $n \times n$ matrix. If M is onto, then M is one-to-one.*

COROLLARY 4. *Let*

$$M : \mathbb{R}^m \to \mathbb{R}^n$$

be given by a matrix which we denote M. Assume that $m < n$. Then M is not onto.

PROOF. By the rank nullity theorem we see that the dimension of the image is less than or equal to m which is strictly less than n. But n is the dimension of $\mathbb{R}^n$. So the image of $M$is a subspace of strictly smaller dimension and hence is not all of $\mathbb{R}^n$.