Public-key Cryptography and elliptic curves

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Cryptography basics

Cryptography is the study of secure communications. Here are some important terms:

- Alice wants to send a message (called the plaintext) to Bob.
- To hide the meaning of the message from others, she encrypts it, transforming the plaintext into the ciphertext
- Bob can decrypt the ciphertext and reveal the plaintext, but a third party (Eve) cannot
- A cipher is an algorithm for performing encryption and/or decryption

Symmetric cryptography

- In a symmetric cipher, the same secret key is used for both encryption and decryption.
- Alice and Bob must share the same key and keep it secret from everyone else
- This is difficult – how do they exchange keys securely?
- Analogy: a locked safe.
    - Alice, Bob have copies the key to open it
    - Each can leave messages there for the other to find

An easy example

Here's a cipher used by Julius Caesar: to encrypt a message, shift each letter N steps forward in the alphabet.

- if $N = 3$, replace every letter with the letter three steps after it in the alphabet.
  - `a` $\rightarrow$ `d`, `b` $\rightarrow$ `e`, etc.
  - '`winrs`' $\rightarrow$ '`zlquv`'
- Decrypt by shifting each letter back N steps
- The secret key is N

An easy example

Why is this cipher so easy to break?

- The key space is small: only 26 possible keys
  - key size $n = \log_2(\text{number of possible keys}) \approx 5$
  - You could easily break this cipher with a brute force attack: try every key until you find the right one.
- The cipher does not hide all the statistical properties of the message
  - Check the frequency with which each letter appears in the ciphertext, compare to the expected frequencies of letters in English language.
  - This is an example of an analytic attack.

How to measure security

Security of a cipher depends on the best known attacks against it and on parameters like key size

- Tradeoff between security and convenience/efficiency
- Assume every practical cipher can be broken given enough time and resources
- If the best known attack is brute force...
  - Key length n bits means 2^n possible keys to try. Impractical for reasonable n
- But if there's a more sophisticated attack with running time polynomial in n, this is probably unsafe regardless of key size
  - Moore's law: computing power per $ grows exponentially over time (for now)
- If a new attack is discovered, the cipher may not be completely ruined; just means bigger keys are necessary

- Today we have much stronger symmetric ciphers available such as AES (Advanced Encryption Standard)
  - Large key space ($n = 128$ or 256). Brute force attacks are effectively impossible
  - Carefully designed to prevent analytic attacks
- But all symmetric ciphers share two inherent weaknesses
  - Alice and Bob must first communicate to share a key, which requires an already secure channel
  - In a network of ≥ 3 people, each pair (e.g. Alice, Bob) needs their own shared key.
    - With N people, that's $N(N-1)/2$ keys in total.

- Public-key cryptography solves these problems
- Basic idea: each person has their own public key and (secret) private key
- Invented* in 1976 by Whitfeld Diffie, Martin Hellman, and Ralph Merkle
  - Invented much earlier by GCHQ (and probably NSA), but not published...
- Analogy: each person has their own locked mailbox with a slot to accept incoming messages
- The mailbox is the public key; the key to open the mailbox is the private key.

Public-key cryptography outline

1. Bob generates both a public key and a private key
   (a) Makes his public key visible to everyone
   (b) Keeps his private key secret
2. Alice encrypts a message using Bob's public key, sends it to Bob
3. Bob can decrypt the message using his private key

- Everyone can send encrypted messages to Bob. Only Bob has the private key to decrypt these messages.
- No secure channel necessary. Alice can send Bob a message without first sharing a secret key.
- In a network of N people, just need N public keys and N private keys.

How are public-key algorithms used?

- Public-key ciphers are slower and less efficient than symmetric ciphers
- Modern secure communication usually works like this:
  1. First use a public-key cipher to securely share a secret key for a symmetric cipher like AES.
  2. Then use the symmetric cipher to actually exchange messages.
- This way we get the best of both worlds!
- Based on mathematical trapdoor functions: easy computations that are hard to reverse.
  - easy-to-compute bijection f with hard-to-compute inverse f^{-1}
- Example: RSA (Rivest, Shamir, Adelman) is based on the problem of factoring a large integer into two primes
  - Easy to multiply $pq = N$
  - But given N, very hard to find p and q

Here's another trapdoor problem. Let p be an odd prime and let b be a generator (primitive root) of the cyclic group $(\mathbb{Z}/p\mathbb{Z})^\times$.

- Given x, it's easy to compute $y = b^x \pmod{p}$ (use "square and multiply" algorithm)
- But given y, it's very hard to compute $x = \log_b y$
  - Different from logarithms in $\mathbb{R}$ where we can use numerical techniques e.g. Newton's method
  - Is there a better way than just trying every value of x?
- The problem of finding x such that $b^x \equiv y \pmod{p}$ is called the discrete logarithm problem (DLP)

Example: Each number in $\mathbb{Z}/31\mathbb{Z}$ appears as 3^x for some x. But there's no easy way to tell when a particular value will appear.

c	$3^x \mod 31$
0	1
1	3
2	9
3	27
4	19
5	26
6	16
7	17
8	20
9	29
10	25

c	$3^x \mod 31$
11	13
12	8
13	24
14	10
15	30
16	28
17	22
18	4
19	12
20	5
21	15

c	$3^x \mod 31$
22	14
23	11
24	2
25	6
26	18
27	23
28	7
29	21
30	1
31	3

Diffie-Hellman key exchange

Suppose Alice and Bob want to communicate using a symmetric cipher like AES.

- They need to share a secret key without anyone else seeing it
- Plan: simultaneously create a key over an insecure channel without sharing private info. This is called key exchange.
- Diffie-Hellman key exchange (DHKE) uses the difficulty of the discrete logarithm problem to keep the key safe from attackers.

DH key exchange algorithm:

- Alice and Bob choose a large prime number p and a primitive root $g \in (\mathbb{Z}/p\mathbb{Z})^\times$. These numbers will be shared publicly.
- Alice chooses a random integer a modulo p to be her private key. She calculates $A = g^a \mod p$, which is her public key.
- Bob chooses a random integer b modulo p to be his private key. He calculates $B = g^b \mod p$, which is his public key.
- Alice and Bob both publish their public keys so everyone can see them. They keep their private keys hidden.

Only Alice knows	Everyone knows	Only Bob knows
a	p, g, A, B	b

Only Alice knows	Everyone knows	Only Bob knows
a	$p,\ g,\ A,\ B$	b

Now it's time to create a shared secret symmetric key.

- Alice calculates $k = B^a \equiv (g^b)^a \equiv g^{ab} \mod m$
- Bob calculates $k = A^b \equiv (g^a)^b \equiv g^{ab} \mod m$
- Now Alice and Bob both know $k = g^{ab}$, which they can use as a shared secret key
- For Eve to find k, she would have to know either a or b, which are the base-g logarithms of A and B modulo p.

A toy example

- $p = 29$, $g = 10$
- Alice chooses $a = 6$ for her private key. She calculates $A = 10^6 \equiv 22 \mod 29$ for her public key
- Bob chooses $b = 21$ for his private key. He calculates $B = 10^{21} \equiv 12 \mod 29$ for his public key
- Alice computes $k = B^a \equiv 12^6 \equiv 28 \mod 29$
- Bob computes $k = A^b \equiv 22^{21} \equiv 28 \mod 29$
- The shared secret key is $k = 28$.

- Note that DHKE cannot actually send an arbitrary message; it only generates a shared secret key.
- There is a similar cipher called ElGamal which is a true encryption/decryption algorithm

Solving the discrete logarithm problem

- In the real world, the key size is probably $n = \log p \approx 1024$ (i.e. p is about 308 digits!)
- Brute force attack: try all 2^{1024} values
  - running time $O(p) = O(2^n)$.
  - would take many, many years even for a supercomputer
- But there are some clever algorithms which speed things up:
  - Pollard rho
  - Pollard Kangaroo
  - Shanks / Baby-step giant-step
- This type of algorithm is called a "Birthday attack"
  - Running time $O(\sqrt{p}) = O\left(2^{\frac{n}{2}}\right)$
  - Better than brute force; equivalent to trying 2^{512} numbers instead of 2^{1024}.
  - Still slow, but doubles the necessary key size for a given security level

Discrete log algorithms: index calculus

There's a much better DLP algorithm called index calculus.

- Uses the fact that many integers modulo p are products of lots of small primes (smooth numbers)
  - Create a factor base of small primes $a_1, a_2, \ldots, a_n$
  - Try to factor $b^k = \prod_{i=1}^{n} a_i^{e_i}$ for different values of k. Each one gives us a linear equation. Need n independent equations.
  - Solving this system gives you the discrete log $x_i = \log_b a_i$ of each prime in the base
  - Now try to factor $b^m y = \prod_{i=1}^{n} a_i^{f_i}$ for some small m
  - $\log y = \sum_{f_i} x_i - m$
- Running time is (more or less) $O\left(e^{c\sqrt[3]{\log p}}\right)$
- IC is strong enough that it forces us to use much larger keys for DHKE

Discrete log in other groups

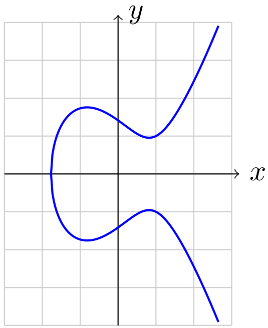
- The DLP we've seen so far is the $(\mathbb{Z}/p\mathbb{Z})^\times$ version.
    - Homomorphism $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ lets us use information about $\mathbb{Z}$ (e.g. prime factorization) to understand $\mathbb{Z}/p\mathbb{Z}$
    - This is why index calculus works – $(\mathbb{Z}/p\mathbb{Z})^\times$ is too easy
- But we can extend the DLP to other finite abelian groups. For instance, an elliptic curve $E(\mathbb{F}_q)$
    - Birthday attacks like Pollard rho will work in any group, but index calculus is specific to $(\mathbb{Z}/p\mathbb{Z})^\times$.
    - This means we can get away with smaller keys!

Definition

An **elliptic curve** E is a smooth plane curve defined by an equation of the form $y^2 = x^3 + ax + b$ for some constants a and b. (Or actually the closure of this curve in projective space)

$E(K)$ is the set of points on this curve defined over the field K.

- $E(\mathbb{C})$ is a compact genus 1 Riemann surface and a complex Lie group
- $E(\mathbb{R})$ is a curve (see right) and a Lie group
- $E(\mathbb{Q})$ is a finitely generated abelian group (Mordell-Weil)
- $E(\mathbb{F}_q)$ is a finite abelian group (cyclic or product of two cyclics)

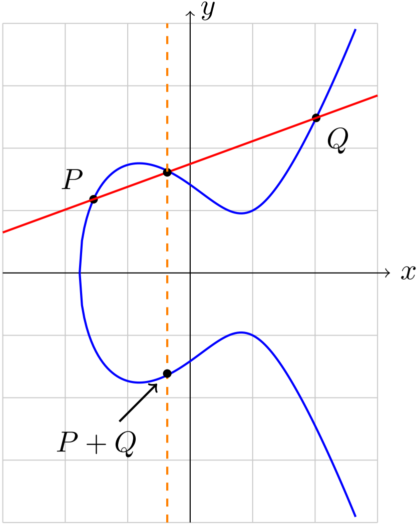


Group structure of an elliptic curve

For group structure on $E(\mathbb{Q}) = \{(x, y) \in \mathbb{Q}^2 : y^2 = x^3 + ax + b\}$ we need:

1. an associative binary operation $+$ such that for any two elements P, Q in G, $P + Q$ is also in $E(\mathbb{Q})$.
2. an identity element I such that $P + I = P$ for all P in $E(\mathbb{Q})$
3. an inverse $-P$ for each element P, such that $P + -P = I$.

- To add $P + Q$:
  - Draw the line $\overline{PQ}$
  - $\overline{PQ}$ intersects the curve at exactly 3 points*
  - Define $P + Q$ to be the reflection across the x-axis of the third intersection point (besides P and Q).
- Easy to prove the following:
  - $P + Q$ is always rational, so $P + Q$ is in $E(\mathbb{Q})$
  - $+$ is associative (and commutative)
- To add $P + P$, draw the tangent to the curve at P



Wait a minute...

Two questions:

1. What happens if you add two points with the same x coordinate?
   - $\overline{PQ}$ is a vertical line
   - Only intersects the curve at P and Q – there's no third point!
2. What is the identity element?
   - We need some point I such that for every point P on the curve, $P + I = P$ and $P + -P = I$.

To answer these questions, we need to add a point at infinity.

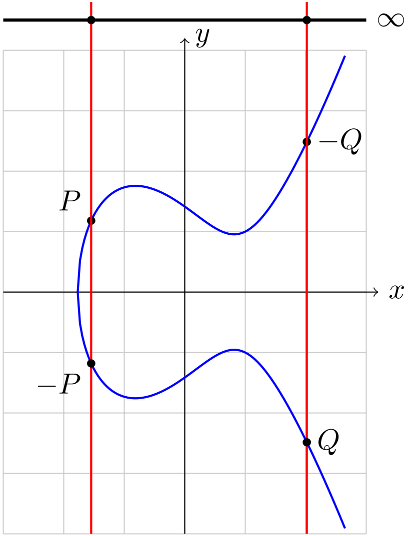
∞

- Think of ∞ as a point that exists infinitely far above (and/or below) the x-axis
- Think of a vertical line $\overline{PQ}$ as passing through three points: P, Q on the curve and ∞.
- ∞ is the identity element
  - To add $P + \infty$, draw a vertical line through P. The line $\overline{P\infty}$ intersects the curve directly above or below P, so $P + \infty = -(-P) = P$.
- The inverse of P is its reflection across the x-axis, $-P$.
  - The line $\overline{P(-P)}$ intersects the curve at $P, -P, and\infty$, so $P + -P = \infty$.

Elliptic curve group operation: identity and inverse

The identity element is ∞

- $P + \infty = P$
- $P + -P = \infty$
- $Q + \infty = Q$
- $Q + -Q = \infty$



Elliptic curve group operation: formula

Let E be an elliptic curve with equation $y^2 = x^3 + ax + b$ and let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be points of $E(\mathbb{Q})$.

- If $P \neq Q$ and $x_1 \neq x_2$, let $s = \frac{y_2 - y_1}{x_2 - x_1}$
- If $P = Q$ and $y_1 \neq 0$, let $s = \frac{3x_1^2 + a}{2y_1}$
- Let $x_3 = s^2 - x_1 - x_2$ and let $y_3 = y_1 - s(x_1 - x_3)$
- Then (x_3, y_3) is the third intersection point of E and $\overline{PQ}$
- Therefore $P + Q = (x_3, -y_3)$.

So you don't have to actually draw lines on a graph to add points. You can just use this formula.

Elliptic curves modulo p

For cryptography we need a finite group, so we use $E(\mathbb{F}_q)$.

- Consider pairs $(x, y) \in \mathbb{F}_q^2$ which satisfy the equation

$$y^2 = x^3 + ax + b$$

- Example: $y^2 \equiv x^3 + x + 6 \mod 7$
  - $(4, 2)$ is a solution because $2^2 \equiv 4^3 + 4 + 6 \equiv 4 \pmod 7$
- The group operation still works. (Use the formulas from the previous slide)
- $E(\mathbb{F}_q)$ is either cyclic or a direct product of two cyclic groups
- Hasse's theorem: $\#E(\mathbb{F}_q) = q + 1 - a_q(E)$ with $|a_q(E)| \leq 2\sqrt{q}$
  - Sato-Tate conjecture: distribution of $a_q(E)$ among curves defined over $\mathbb{F}_q$

The Elliptic Curve Discrete Log Problem

- If you have a number n and a point P on the curve, it's easy to add P to itself n times and find the point nP
  - Fast algorithm "double and add" (analogous to "square and multiply" for exponentiation mod m)
- But, if you have P and an arbitrary point Q, how do you find a number n such that P added to itself n times is Q?
  - If you keep adding P you'll eventually hit every point on the curve, but in an unpredictable order.
- This is the discrete log problem,in the group $E(\mathbb{F}_q)$ instead of $(\mathbb{Z}/p\mathbb{Z})^{\times}$. We call it ECDLP
- ECDH is a version of Diffie-Helman key exchange that uses the $E(\mathbb{F}_q)$ version of the discrete logarithm problem.

Elliptic Curve Diffie-Hellman

Alice and Bob want to securely generate a shared secret key

- They agree on an elliptic curve E, a prime p, and a base point P on E. These things are all shared publicly.
- Alice chooses a random positive integer a to be her private key. She adds P to itself a times to get a point $A = aP$ on E. This is her public key.
- Bob chooses a random positive integer b to be his private key. He adds P to itself b times to get a point $B = bP$ on E. This is his public key.
- Alice and Bob publish their public keys, but keep their private keys secret.

Elliptic Curve Diffie-Hellman

- Alice adds B to itself a times, getting $k = a(bP) = (ab)P$.
- Bob adds A to itself b times, getting $k = b(aP) = (ab)P$.
- Now Alice and Bob both know $k = (ab)P$, which they can use as a shared secret key.
- For a third person to find k, they would have to compute a or b, i.e. the discrete log of A or B in $E(\mathbb{F}_p)$.

Why use ECC?

- Birthday attacks work for DLP in any group, including $E(\mathbb{F}_p)$: Pollard rho, Kangaroo, etc
- But index calculus (mostly) only works for $(\mathbb{Z}/p\mathbb{Z})^\times$
  - IC relies on using information about $\mathbb{Z}$ (prime factorization)
  - For supersingular elliptic curves there is a version of index calculus. But we avoid this by not using those curves
- Best known attacks (for general curves) are of the Rho/Kangaroo/BSGS type, which are much slower
- Same security level as DH with much smaller keys!

Choosing a curve

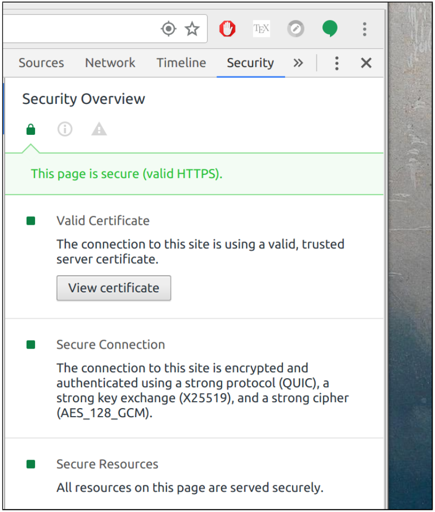
Security and efficiency of ECC depends on choice of curve

- Supersingular curves (lots of endomorphisms) are bad
- $E(\mathbb{F}_p)$ where p has small order k modulo $\#E(\mathbb{F}_p)$ are bad
  - Use Tate pairing to reduce to DLP in $(\mathbb{F}_{p^k})^\times$
- Curves over $\mathbb{F}_{2^k}$ have very fast arithmetic, but there are good specialized algorithms for this case
- Some curves may have hidden weaknesses we can't see
  - 2013: Snowden leaks reveal that `Dual_EC_DRBG` random number generator has a backdoor created by the NSA
  - 2015: NSA recommends phasing out ECC-based crypto algorithms (why?)
- Example of a good curve: Curve25519 (Daniel Bernstein)

$$y^2 = x^3 + 486662x^2 + x \qquad p = 2^{255} - 19$$

ECC in the real world

- HTTPS often uses key exchange with Curve25519
- Sony PS3 used ECDSA to sign executables (oops)
- Online messaging protocols
- And much more!



Thanks!

Further reading:

- *Understanding Cryptography* by C. Paar, J. Pelzl
  - Good simple textbook on modern crypto algorithms. Written for engineers, no hardcore number theory
- *The Arithmetic of Elliptic Curves* by Joseph H. Silverman
  - Standard graduate text on elliptic curves.