Zeta Functions of a Family of Quaternion Extensions of $\mathbb{F}_p(t)$

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Order of vanishing of zeta functions at $s = 1/2$

An L-function is a meromorphic function on $\mathbb{C}$ derived from a Dirichlet series associated to some object such as a number field, variety, or representation.

- Should satisfy some kind of functional equation w.r.t. a vertical line $\operatorname{Re} s = k$:

$$\Lambda(s) = W\Lambda(k - s)$$

  - $\Lambda(s)$ is the 'completed' L-function
  - $W \in \mathbb{C}$ is the root number, with $|W| = 1$.
- Heuristic: an L-function should vanish at the lowest order compatible with its root number.
  - If an L-function breaks this rule, there should be a reason.

Order of vanishing of zeta functions at $s = 1/2$

- Idea: look for objects with L-functions which vanish at an order higher than required by their root numbers.
- For fields with Galois group Q_8, we know the root number is ± 1, so the expected order of vanishing is 0 or 1.
- Omar [2012] computed order of vanishing at $s = 1/2$ of L-functions of octic number fields with Galois group Q_8.
  - In all the fields Omar checked, order of vanishing is 0 or 1
- What about function fields? Let's try quaternion extensions of $\mathbb{F}_q(t)$ and see if we can find some where the zeta function vanishes to order > 1.
- We found an infinite family of quaternion function fields for which $\mathrm{ord}_{s=1/2}\,\zeta(s) = 2$
- This also leads to a computable example for a theorem of Ramachandran on motivic interpretation of $\mathrm{ord}_{s=1/2}\,\zeta(s)$ for varieties over $\mathbb{F}_p^2$.

Theorem (Witt's Criterion, 1936)

Let $K(\sqrt{a},\sqrt{b})$ be a biquadratic extension. Then $K(\sqrt{a},\sqrt{b})/K$ can be embedded in a quaternion extension if and only if the quadratic forms $aX^2+bY^2+abZ^2$ and $U^2+V^2+W^2$ are K-equivalent.

Furthermore, if $P=(p_{ij})\in K^{3\times 3}$ such that

$$P^T\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & ab \end{bmatrix}P=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then the quaternion extensions of K containing $K(\sqrt{a},\sqrt{b})$ are of the form

$$K\left(\sqrt{r\left(1+p_{11}\sqrt{a}+p_{22}\sqrt{b}+p_{33}\sqrt{ab}\right)}\right),\quad r\in K^{\times}.$$

- Hasse-Minkowski: two quadratic forms are equivalent over a global field K iff they are locally equivalent at every place.
  - So $K(\sqrt{a}, \sqrt{b})$ extends to a quaternion extension iff $aX^2 + bY^2 + abZ^2$ is locally equivalent to $U^2 + V^2 + W^2$ at every place $\mathfrak{p}$ of K.
  - If true, we call (a, b) a Witt pair.
- The hard part is finding the transformation matrix P. We need this to actually generate quaternionic fields:

$$K\left(\sqrt{r\left(1 + p_{11}\sqrt{a} + p_{22}\sqrt{b} + p_{33}\sqrt{ab}\right)}\right)$$

- We used Magma to find Witt pairs and compute (random) P for each.
- Different P will generate different fields (twists) lying above $K(\sqrt{a}, \sqrt{b})$.

Here's one case where we can compute $\operatorname{ord}_{s=1/2}\zeta(s)$ explicitly.

- Let $K = \mathbb{F}_p(t)$ where $p \equiv 5 \pmod 8$.
  - $\mathbb{F}_p$ has simple quadratic reciprocity and a 4th root of unity (i) but no 8th RoU.
  - Here (a, b) is a Witt pair iff a is square modulo b.
- Let $a = t + w^2$ for some $w \in \mathbb{F}_p^\times$ and let $b = t$. We want to construct a quaternion extension of K containing $K(\sqrt{a}, \sqrt{b})$.
- Here's an explicit transformation matrix P for this case:

$$P = \frac{1}{2wab^2}\begin{bmatrix} 0 & 2iw^2ab & 2wab \\ b(ab^2 - w^2) & a(ab^2 + w^2) & -iw(ab^2 + w^2) \\ -b(ab^2 + w^2) & ia(ab^2 - w^2) & w(ab^2 - w^2) \end{bmatrix}$$

This satisfies $P^T AP = I_{3\times 3}$ where A is the matrix of $aX^2 + bY^2 + abZ^2$.

Using this matrix, we can construct a quaternion extension of K containing $K(\sqrt{a}, \sqrt{b})$:

$$L = K\left(\sqrt{1 + \frac{ab^2 + w^2}{2wb^2}\sqrt{b} + \frac{ab^2 - w^2}{2ab^2}\sqrt{ab}}\right).$$

Now we want to prove that the zeta function $\zeta(s)$ of this field has $\operatorname{ord}_{s=1/2}\zeta(s) = 2$.

The biquadratic field $K(\sqrt{a}, \sqrt{b})$ has genus zero:

- Defining polynomial of $K(\sqrt{a}, \sqrt{b})$ over $\mathbb{F}_p(t)$ is $x^4 - 2(2t + w^2)x^2 + w^4$
- $t = (x^4 - 2w^2x^2 + w^4)/(4x^2)$, so actually this field is just $\mathbb{F}_p(x)$ – the field of functions of a projective line.

Rewrite primitive element for L in terms of x:

$$\begin{aligned} L &= \mathbb{F}_p(x)\left(\sqrt{\frac{(x^6 + 2wx^5 + w^2x^4 + 8wx^3 - w^4x^2 - 2w^5x - w^6)^2}{16wx^3(x-w)(x+w)^3(x^2+w^2)}}\right) \\ &= \mathbb{F}_p(x)\left(\sqrt{wx(x^4 - w^4)}\right) \end{aligned}$$

This is the field of functions of the curve $y^2 = wx(x^4 - w^4)$. Or with a quick change of variables, the curve $Y^2 = X(X^4 - 1)$. This is a hyperelliptic curve of genus 2.

So L is the field of functions of a genus-2 curve $C_1 : Y^2 = X(X^4 - 1)$. The zeta function is then

$$\zeta(s) = \exp\left(\sum_{k=1}^{\infty} \frac{\#C(\mathbb{F}_{p^k})}{k} (p^{-s})^k\right) = \frac{P(q^{-s})}{(1 - p^{-s})(1 - p^{1-s})}$$

where $P(T) \in \mathbb{Z}[T]$ has degree 4. To find $\zeta(s)$, we start by computing $\#C(\mathbb{F}_p)$.

- Recall $p \equiv 5 \mod 8$. Let $i \in \mathbb{F}_p$ be a square root of -1. Note i is not a square.
- let $h(X) = X(X^4 - 1) = X(X+1)(X-1)(X+i)(X-i)$. There are 5 roots, each of which gives one point on the curve.
- For all X outside the roots of h, $h(iX) = ih(X)$. Since i is not square, exactly one of $h(X)$, $h(iX)$ is a square.
- So half of the non-roots yield 2 points each, the other half yield zero points.
- Total number of projective points is $\#C_1(\mathbb{F}_p) = p + 1$.

However we can't use this trick for $\mathbb{F}_{p^2}$ because in this field, i is a square.

Computing $\#C(\mathbb{F}_{p^2})$

Let C_2 be the curve $Y^2 = X^6 + 5X^4 - 5X^4 - 1$. The curves C_1 and C_2 are birationally equivalent over $\mathbb{F}_{p^4}$, but not over $\mathbb{F}_{p^2}$. So in $\mathbb{F}_{p^2}$ the two differ by a quadratic twist.

- We have two natural maps from C_2 to the supersingular elliptic curve $E : Y^2 = X^3 + 5X^2 - 5X - 1$.
- These extend to maps on the Jacobian, meaning $J(C_2) \simeq E \times E$.
- Cassels, Flynn:

$$\#J(C)(\mathbb{F}_p) = \frac{1}{2}\#C(\mathbb{F}_{p^2}) + \frac{1}{2}(\#C(\mathbb{F}_p))^2 - p$$

  We can use this to compute $\#C_2(\mathbb{F}_{p^2}) = p^2 + 4p + 1$.
- Since C_2 is a quadratic twist of C_1, this means $\#C_1(\mathbb{F}_{p^2}) = p^2 - 4p + 1$.

We computed $\#C_1(\mathbb{F}_p) = p + 1$ and $\#C_1(\mathbb{F}_{p^2}) = p^2 - 4p + 1$. We then find that the zeta function must be

$$\zeta(s) = \frac{(1 - p^{1-2s})^2}{(1 - p^{-s})(1 - p^{1-s})}$$

and so clearly $\operatorname{ord}_{s=1/2} \zeta(s) = 2$.

Theorem

Let $K = \mathbb{F}_p(t)$ where $p \equiv 5 \pmod 8$ and let $w \in \mathbb{F}_p^\times$. Let $a = t + w^2$ and $b = t$. Let $\alpha^2 = 1 + \frac{ab^2+w^2}{2wb^2}\sqrt{b} + \frac{ab^2-w^2}{2ab^2}\sqrt{ab}$ and let $L = K(\alpha)$. Then

(1) $\mathrm{Gal}(L/K) = Q_8$, *meaning the root number of $\zeta(K, s)$ is ± 1*

(2) *The genus is $g(L) = 2$*

(3) $\operatorname{ord}_{s=1/2} \zeta(K, s) = 2$.

- Ramachandran (2005): Let X be a smooth projective variety over $\mathbb{F}_{p^2}$. Let E be a certain supersingular elliptic curve over $\mathbb{F}_{p^2}$. Then

$$-2\,\mathrm{ord}_{s=1/2}\,\zeta(X,s) = \sum(-1)^j j r_j$$

  where r_j is the rank of the Weil-ètale cohomology group $H^j_W(X,E)$.
  - In fact, $r_0 = r_1 = 2\,\mathrm{ord}_{s=1/2}\,\zeta(X,s)$ and all others are 0.
- If X is a curve defined over $\mathbb{F}_p$, let $\zeta(X,s)$ be the zeta function of $X/\mathbb{F}_p$ and let $\zeta_2(s)$ be the zeta function of $X/\mathbb{F}_{p^2}$. Then $\mathrm{ord}_{s=1/2}\,\zeta_2(s) = 2\,\mathrm{ord}_{s=1/2}\,\zeta(s)$.
- Our family of curves provides a computable example where $\mathrm{ord}_{s=1/2}\,\zeta_2(s) = 4$ is higher than required by the root number.

- For this same type of extension but with $p \equiv 1 \mod 8$, we found experimentally that $\mathrm{ord}_{s=1/2}\,\zeta(s) = 0$. Not proven because counting points is a lot harder here.
- For Witt pairs where $\deg a, \deg b > 1$, we found many quaternion fields with $\mathrm{ord}_{s=1/2}\,\zeta(s) > 1$.

p	a	b	g	$\mathrm{ord}_{s=1/2}$
5	t^2+t+2	$t+2$	7	4
5	t^2+4t+3	$2t^2+4t+3$	7	4
5	t^2+4t+3	$t^2+4*t+2$	7	4
5	t^2+4t	$2t+4$	7	4

Thanks!

- J. Cassels and E. Flynn, *Prolegomena to a Middlebrow Arithmetic of Curves of Genus 2*
- S. Omar, *On Artin L-Functions for Octic Quaternion Fields*
- C. Jensen, A. Ledet, and N. Yui, *Generic Polynomials: Constructive Aspects of the Inverse Galois Problem*
- N. Ramachandran, *Values of Zeta Functions at $s = 1/2$*