Riemann's Zeta Function and the Prime Number Theorem

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Let's begin with the Basel problem, first posed in 1644 by Mengoli.
Find the sum of the following infinite series:

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- First solved* by Euler in 1735

$$\sin(\pi x) = \pi x - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} + \ldots$$

$$\sin(\pi x) = \pi x(1 - x^2)\left(1 - \frac{x^2}{4}\right)\left(1 - \frac{x^2}{9}\right)\cdots$$

$$= \pi x + \pi x^3 \left(1 + \frac{1}{4} + \frac{1}{9} + \ldots\right) + \pi x^5 (\cdots) + \ldots$$

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- Euler's proof was incomplete; he assumed that a power series can be written as an infinite product of linear polynomials.
  - It's actually true in this case, but not always
- There are about a dozen other crazy ways to prove this.

Riemann's zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

- This series converges for real numbers $s > 1$, diverges for real numbers $s \leq 1$
- $s = 1$ is the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$
- The Basel problem is really asking "what is the value of $\zeta(2)$?" complex function

A prime number is a natural number whose only divisors are 1 and itself.

Here's the simplest connection between the zeta function and prime numbers:

$$\prod_p \left(\frac{1}{1-p^{-s}}\right) = \prod_p \left(\sum_{k=0}^{\infty} (p^{-s})^k\right) = \prod_p \left(\sum_{k=0}^{\infty} (p^k)^{-s}\right)$$

$$= \left(1 + 2^{-s} + (2^2)^{-s} + \dots\right) \left(1 + 3^{-s} + (3^2)^{-s} + \dots\right) \cdots$$

$$= 1^{-s} + 2^{-s} + (2^2)^{-s} + 5^{-s} + (2 \cdot 3)^{-s} + 7^{-s} + (2^3)^{-s} + \dots$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s).$$

Theorem

Let $X = \{x_1, x_2, \ldots, x_s\}$ be a set of s randomly chosen integers. Let P be the probability that $\gcd(X) > 1$. Then $P = 1/\zeta(s)$.

- Let p be a prime. $P(x_i \text{ is divisible by } p) = 1/p$.
- $P(s \text{ random numbers are all divisible by } p) = (1/p)^s$
- $P(\text{at least one } x_i \text{ is not divisible by } p) = 1 - p^{-s}$
- Probability that there is no prime which divides all x_i is the product of this expression over all primes:

$$\begin{aligned} P &= \prod_p \left(1 - p^{-s}\right) \\ &= \frac{1}{\zeta(s)}. \end{aligned}$$

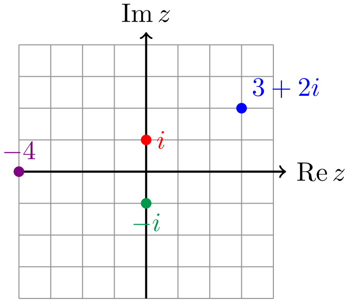
Two minute intro to complex numbers part 1

$\zeta(s)$ was studied first by Euler as a real function. Riemann was the first to view it as a complex function.

Definition

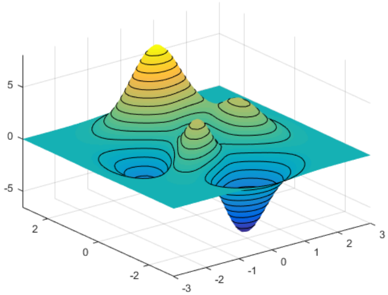
Let i be an imaginary number with $i^2 = -1$. Complex numbers are expressions of the form $a + bi$ where $a, b \in \mathbb{R}$. We call this set $\mathbb{C}$.

- Let $z = a + bi$. The real part of z is $\mathrm{Re}(z) = a$ and the imaginary part is $\mathrm{Im}(z) = b$.
- View these as points in the complex plane, i.e. $(x, y) = (\mathrm{Re}\, z, \mathrm{Im}\, z)$.
- Magnitude of z is $|z| = \sqrt{a^2 + b^2}$ (distance from z to the origin)



Two minute intro to complex numbers part 2

- You can add, subtract, multiply, and divide complex numbers
- You can make a function whose input and output are complex numbers, e.g. $f(z) = z^2$
  - $f(2+i) = (2+i)^2 = 2^2 + i^2 + 2*2*i = 3 + 4i$
- Hard to graph because both domain and range are 2 dimensional



MathWorks

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - p^{-s}\right)^{-1}$$

Riemann described the complex zeta function in his 1859 paper *Über die Anzahl der Primzahlen unter einer gegebenen Grösse*.

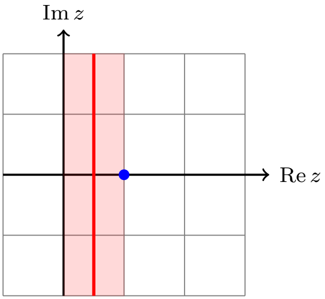
- $\zeta(s)$ converges for all $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$
- There is a meromorphic continuation of $\zeta(s)$ to the rest of $\mathbb{C}$ (with a simple pole at $s = 1$)

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^s}{e^z - 1} \frac{dz}{z}$$

- $\zeta(s)$ satisfies a functional equation:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$$

The functional equation $\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$ relates values on opposite sides of the critical line $\mathrm{Re}\, z = 1/2$:



It's known that all* zeros of $\zeta(s)$ lie in the critical strip $0 < \mathrm{Re}\, z < 1$.

Conjecture (Riemann hypothesis)

All nontrivial zeros of $\zeta(s)$ *have real part* $1/2$*.*

- There are trivial zeroes at all negative even integers
- This is one of the most famous unsolved problems in all of mathematics. If you can solve it, you'll get $1 million and probably a Fields medal.
- Why is this important?
  - $\zeta(s)$ encodes lots of deep information about $\mathbb{Z}$
  - Meromorphic complex functions are largely defined by the locations and orders of their zeros and poles.
  - Knowing the zeros is important for computing contour integrals. (more on this later)

- Number theory: branch of mathematics that studies the integers, $\mathbb{Z}$
- However, in order to understand $\mathbb{Z}$ we often have to work with other mathematical objects:
  - $\mathbb{Q}$ (rational numbers) and $\mathbb{C}$ (complex numbers)
  - Finite fields $\mathbb{F}_q$ (modular arithmetic)
  - Rings and fields of polynomials, e.g. $\mathbb{Z}[t]$, $\mathbb{F}_q(t)$
  - Geometric objects like algebraic curves
- Goal is to understand arithmetic in $\mathbb{Z}$. A major part of this is understanding how composite numbers break down into primes.
  - Another major part is understanding integer solutions to equations like $x^2 - ny^2 = 1$

- The first few prime numbers are $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, \ldots$
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- Hadamard and de la Vallée-Poussin, 1896: Prime Number Theorem (with an assist from Riemann)

Theorem (PNT)

Let $\pi(x) = \#\{p \in \mathbb{Z} : p \leq x\}$ *be the prime counting function. As* $x \to \infty$, $\pi(x) \sim x/\log x$. *That is,*

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1.$$

Instead of working with $\pi(x)$ directly, we'll use the von Mangoldt function

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \\ 0 & \text{otherwise} \end{cases}$$

and the Chebyshev function

$$\psi(x) = \sum_{n \leq x} \Lambda(n).$$

The PNT is equivalent to proving that $\lim_{x\to\infty} \frac{\psi(x)}{x} = 1$.

Here's where $\zeta(s)$ comes in: it turns out that

$$-\frac{d}{ds}\log\zeta(s) = \sum_{n=1}^{\infty} \Lambda(n) n^{-s}.$$

Then you can do some analysis and prove the following equation:

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log(2\pi)$$

summing over the zeros ρ of $\zeta(s)$. If you know where the ρ are, you can prove that this is $\psi(x) = x - (\text{lower order terms})$.

Or compute a certain contour integral in the complex plane. Again, need to know the zeros of $\zeta(s)$.

Proving the Prime Number Theorem

- Riemann stated his hypothesis in 1859. For decades afterwards, mathematicians knew that proving the RH would prove the PNT.
- 1896: Hadamard and de la Vallée-Poussin (independently) proved that all nontrivial zeros of $\zeta(s)$ lie in the critical strip $0 < \mathrm{Re}\, z < 1$.
  - This is weaker than the RH, but it's enough to prove the PNT
- An even better estimate is $\pi(x) = \int_2^t \frac{1}{\log t}\, dt$.

Prime numbers post-PNT

- There are still many unanswered questions about the distribution of primes in $\mathbb{Z}$, e.g. twin prime conjecture
- For the past few decades, prime numbers (and number theory in general) have become important because of their use in crypto algorithms
- There are other zeta functions (and L-functions) for other types of mathematical objects such as number fields, varieties, representations...
    - Extended/Generalized Riemann Hypothesis (ERH/GRH): RH for other types of zeta functions
    - Many number theory papers prove important results if the GRH is true!

End

Further reading:

- J. Derbyshire, *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics*
- H.M. Edwards, *Riemann's Zeta Function*