1 Waves with a source

Consider the inhomogeneous wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), & -\infty < x < \infty, t > 0, \\ u(x,0) = \phi(x), & u_t(x,0) = \psi(x), \end{cases} \tag{1.1}$$

where $f(x,t)$, $\phi(x)$ and $\psi(x)$ are arbitrary given functions. Similar to the inhomogeneous heat equation, the right hand side of the equation, $f(x,t)$, is called the *source* term. In the case of the string vibrations this term measures the external force (per unit mass) applied on the string, and the equation again arises from Newton's second law, in which one now also has a nonzero external force.

As was done for the inhomogeneous heat equation, we can use the superposition principle to break problem (1.1) into two simpler ones:

$$\begin{cases} u^h_{tt} - c^2 u^h_{xx} = 0, \\ u^h(x,0) = \phi(x), \quad u^h_t(x,t) = \psi(x), \end{cases}$$

and

$$\begin{cases} u^p_{tt} - c^2 u^p_{xx} = f(x,t), \\ u^p(x,0) = 0, \quad u^p_t(x,t) = 0. \end{cases} \tag{1.2}$$

Obviously, $\mathbf{u} = \mathbf{u^h} + \mathbf{u^p}$ will solve the original problem (1.1). The function u^h solves the homogeneous equation, so it is given by d'Alambert's formula. Thus, we only need to solve the inhomogeneous equation with zero data, i.e. problem (1.2). We will show that the solution to the original IVP (1.1) is

$$\mathbf{u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds.} \tag{1.3}$$

The first two terms in the above formula come from d'Alambert's formula for the homogeneous solution u^h, so to prove formula (1.3), it suffices to show that the solution to the IVP (1.2) is

$$u^p(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds. \tag{1.4}$$

For simplicity, we will seize specifying the superscript and write $u = u^p$ (this corresponds to the assumption $\phi(x) \equiv \psi(x) \equiv 0$, which is the only remaining case to solve).

Recall that we have already solved inhomogeneous hyperbolic equations by the method of characteristics, which we will apply to the inhomogeneous wave equation as well. The change of variables into the characteristic variables and back are given by the following formulas

$$\begin{cases} \xi = x + ct, \\ \eta = x - ct, \end{cases} \qquad \begin{cases} t = \frac{\xi - \eta}{2c}, \\ x = \frac{\xi + \eta}{2}. \end{cases} \tag{1.5}$$

To write the equation in the characteristic variables, we compute u_{tt} and u_{xx} in terms of (ξ,η) using the chain rule.

$$u_t = cu_\xi - cu_\eta, \qquad u_x = u_\xi + u_\eta,$$

$$u_{tt} = c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta}, \qquad u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta},$$

so

$$u_{tt} - c^2 u_{xx} = -4c^2 u_{\xi\eta}. \tag{1.6}$$

Using (1.6), we can rewrite the inhomogeneous wave equation in terms of the characteristic variables as

$$u_{\xi\eta} = -\frac{1}{4c^2} f(\xi,\eta). \tag{1.7}$$

To solve this equation, we need to successively integrate in terms of η and then ξ. Recall that in previous examples of inhomogeneous hyperbolic equations we performed these integrations explicitly, then changed the variables back to (x,t), and determined the integration constants from the initial conditions. In our present case, however, we would like to obtain a formula for the general function f, so explicit integration is not an option. Thus, to determine the constants of integration, we need to rewrite the initial conditions in terms of the characteristic variables.

Notice that from (1.5), $t = 0$ is equivalent to $(\xi - \eta)/2c = 0$, or $\xi = \eta$. The initial conditions of (1.2) then imply

$$u(\xi,\xi) = 0,$$

$$cu_\xi(\xi,\xi) - cu_\eta(\xi,\xi) = 0,$$

$$u_\xi(\xi,\xi) + u_\eta(\xi,\xi) = 0,$$

where the last identity is equivalent to the identity $u_x(x,0) = 0$, which can be obtained by differentiating the first initial condition of (1.2). From the last two conditions above, it is clear that $u_\xi(\xi,\xi) = u_\eta(\xi,\xi) = 0$, so the initial conditions in terms of the characteristic variables are

$$u(\xi,\xi) = u_\xi(\xi,\xi) = u_\eta(\xi,\xi) = 0. \tag{1.8}$$

Now fix a point (x_0,t_0) for which we will show formula (1.4). This point has the coordinates (ξ_0,η_0) in the

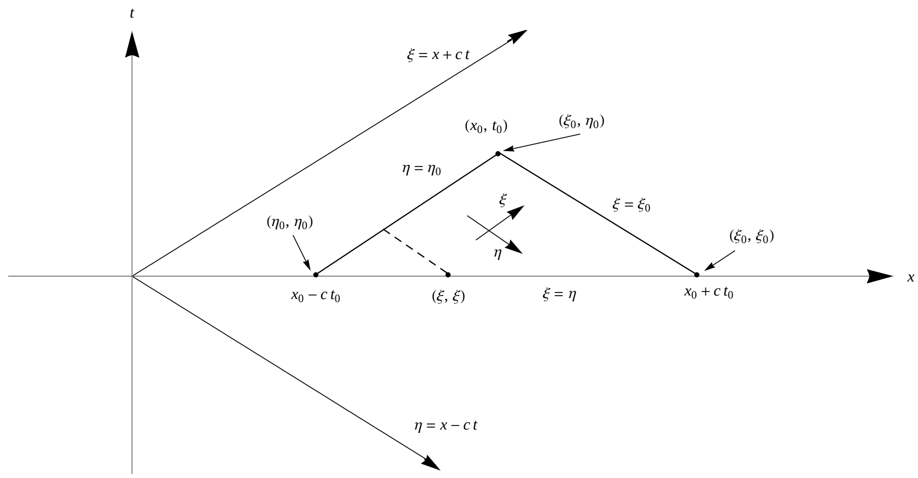

Figure 1.1: The triangle of dependence of the point (x_0,t_0).

characteristic variables. To find the value of the solution at this point, we first integrate equation (1.7) in terms of η from ξ to η_0

$$\int_{\xi}^{\eta_0} u_{\xi\eta} d\eta = -\frac{1}{4c^2}\int_{\xi}^{\eta_0} f(\xi,\eta)d\eta.$$

But

$$\int_{\xi}^{\eta_0} u_{\xi\eta} d\eta = u_\xi(\xi,\eta_0) - u_\xi(\xi,\xi) = u_\xi(\xi,\eta_0)$$

due to (1.8) (this is precisely the reason for the choice of the lower limit), so we have

$$u_\xi(\xi,\eta_0) = \frac{1}{4c^2}\int_{\eta_0}^{\xi} f(\xi,\eta)d\eta.$$

Integrating this identity with respect to ξ from η_0 to ξ_0 gives

$$\int_{\eta_0}^{\xi_0} u_\xi(\xi,\eta_0)d\xi = \frac{1}{4c^2}\int_{\eta_0}^{\xi_0}\int_{\eta_0}^{\xi} f(\xi,\eta)d\eta d\xi.$$

Similar to the previous integral,

$$\int_{\eta_0}^{\xi_0} u_\xi(\xi,\eta_0)d\xi = u(\xi_0,\eta_0) - u_\xi(\eta_0,\eta_0) = u(\xi_0,\eta_0)$$

due to (1.8). We then have

$$u(\xi_0,\eta_0) = \frac{1}{4c^2}\int_{\eta_0}^{\xi_0}\int_{\eta_0}^{\xi} f(\xi,\eta)d\eta d\xi = \frac{1}{4c^2}\iint_{\triangle} f(\xi,\eta)d\xi d\eta, \tag{1.9}$$

where the double integral is taken over the triangle of dependence of the point (x_0,t_0), as depicted in Figure

1.1. Using the change of variables (1.5), and computing the Jacobian,

$$J=\frac{\partial(\xi,\eta)}{\partial(x,t)}=\begin{vmatrix} 1 & c \\ 1 & -c \end{vmatrix}=-2c,$$

we can transform the double integral in (1.9) to a double integral in terms of the (x,t) variables to get

$$u(x_0,t_0)=\frac{1}{4c^2}\iint_{\triangle} f(x,t)|J|dxdt=\frac{1}{2c}\iint_{\triangle} f(x,t)dxdt.$$

Finally, rewriting the last double integral as an iterated integral, we will arrive at formula (1.4). This finishes the proof that (1.3) is the unique solution of the IVP (1.1). One can alternatively show that formula (1.3) gives the solution by directly substituting it into (1.1), which is left as a homework problem.

Example 1.1. *Solve the inhomogeneous wave IVP*

$$\begin{cases} u_{tt}-c^2u_{xx}=e^x, \\ u(x,0)=u_t(x,0)=0. \end{cases}$$

Using formula (1.3) *with* $\phi=\psi=0$*, we get*

$$\begin{aligned} u(x,t)&=\frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)} e^y dyds=\frac{1}{2c}\int_0^t [e^{x+c(t-s)}-e^{x-c(t-s)}]ds \\ &=\frac{e^x}{2c}\left(-\frac{1}{c}e^{c(t-s)}\Big|_0^t-\frac{1}{c}e^{-c(t-s)}\Big|_0^t\right)=\frac{e^x}{2c^2}(e^{ct}+e^{-ct}-2). \end{aligned}$$

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