

Erratum: Chapter 1, Theorem 4.2

The proof of Theorem 4.2 in the book is incorrect: it is neither proved nor true that the ϕ_k are increasing. However a small modification of the proof makes it all ok.

Theorem 0.1 *Let f be a nonnegative measurable function. Then there exists a sequence of simple functions $\phi_k(x)$ such that $\phi_k(x) \leq \phi_{k+1}(x)$ and $\lim_{k \rightarrow \infty} \phi_k(x) = f(x)$ for all x .*

Proof: Let Q_k be the cube of side length k centered at the origin and let

$$F_k(x) = \begin{cases} f(x) & \text{if } x \in Q_k \text{ and } f(x) \leq k \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Then $F_k(x) \leq F_{k+1}(x)$ and $\lim_{k \rightarrow \infty} F_k(x) = f(x)$.

We divide the range of F_k , $[0, k]$ into $k2^k$ intervals of size $1/2^k$:

$$E_{k,j} = \left\{ x : \frac{j-1}{2^k} < F_k(x) \leq \frac{j}{2^k} \right\} \quad j = 1, 2, \dots, k2^k.$$

and set

$$\phi_k(x) = \sum_{j=1}^{k2^k} \frac{j-1}{2^k} \chi_{E_{k,j}}.$$

We have $\phi_k(x) \leq F_k(x)$ and $F_k(x) - \phi_k(x) \leq 1/2^k$. Therefore $f(x) = \lim_{k \rightarrow \infty} F_k(x) = \lim_{k \rightarrow \infty} \phi_k(x)$. It remains to show that $\phi_k(x)$ is increasing. To see this note that one obtains the $E_{k+1,j}$'s by dividing each $E_{k,j}$ into two pieces and adding 2^{k+1} pieces to cover $(k, k+1]$. We have $E_{k+1,j} = E_{k+1,2j-1} \cup E_{k,2j}$ for $j = 1, \dots, k2^k$. This property implies that $\phi_k \leq \phi_{k+1}$. ■