Sketch of a solution to Problem 12 Section 1.1

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We start with $\Longrightarrow$: ie. we want to prove that is $P \in \mathcal{F}$ satisfies the properties (a) and (b) then $\leq$ -as defined in the problem- satisfies properties (O1)-(O5).

Proof of (O1) : Let $x, y \in \mathcal{F}$ by the definition of $\leq$ we need to show that that either $(y-x) \in P$ or $(x-y) \in P$ or $y = x$.

Since $\mathcal{F}$ is a field, if x and y are in $\mathcal{F}$, then $-x$ and hence $y + (-x) = y - x$ are in $\mathcal{F}$. Let's call $z = y - x$. By (a) then we know that either $z \in P$, $-z \in P$ or $z = 0$. That is either $(y-x) \in P$, $-(y-x) \in P$ or $y - x = 0$.

But additive inverses are unique so by P1) P2) and P4 we have that $-(y-x) = (x-y)$. and by P1) P3) and P4) we have that $y - x = 0$ implies $y = x$

Hence all in all we have that either $(y-x) \in P$, $(x-y) \in P$ or $y = x$ as desired.

Proof of (O2): Suppose that both $x \leq y$ and $y \leq x$. To say $x \leq y$ means that $(y-x) \in P$ or $x = y$. And to say that $y \leq x$ means that $(x-y) \in P$ or $x = y$.

But if we suppose that $(y-x) \in P$; then $-(y-x) = (x-y)$ (as we showed above) is not in P. And hence we must have $x = y$. But $x = y$ means $y - x = 0$ which contradicts (a) since we are assuming $(y-x) \in P$.

A similar argument gives a contradiction if we suppose that $(x-y) \in P$.

Therefore we must have that $x = y$.

Proof of (O3): If $x \leq y$ then $(y-x) \in P$ or $x = y$. If $y \leq z$ then $(z-y) \in P$ or $y = z$.

Now if $(y-x) \in P$ and $(z-y) \in P$ then $(z-y)+(y-x) \in P$ by (b). Therefore $z + (-y+y) - x \in P$ by P2) and $z + 0 - x \in P$ by P4). By P3) we then have that $z - x \in P$. Then $x \leq z$.

If $(y-x) \in P$ and $y = z$ then $y - x = z - x$ so $x \leq z$.

If $(z-y) \in P$ and $x = y$ then $z - y = z - x$ so $x \leq z$.

If $z = y$ and $x = y$ then $z = x$, so $x \leq z$.

Proof of (04): If $x \leq y$ then either $(y-x) \in P$ or $x = y$. If $(y-x) \in P$, $y - x + 0 \in P$. Since $z + (-z) = 0$ we then have that $(y-x) + (z + (-z)) \in P$. By P1) and P2) then $(y+z) - (x+z) \in P$ which means that $x + z \leq y + z$ as desired.

If $x = y$ then $x + z = y + z$; therefore $x + z \leq y + z$ once again.

Proof of (O5): If $x \leq y$ and $0 \leq z$ then either $(y-x) \in P$ or $x = y$ and either $z = 0$ or $z \in P$.

If $z = 0$, then $xz = 0 = yz$ (by homework problem 4- also proved in class); so in this case $xz \leq yz$.

If $x = y$ then $xz = yz$ so $xz \leq yz$.

If $z \in P$ and $(y-x) \in P$, then $z(y-x) \in P$ by (b). Now, by P9) $zy - zx \in P$. By P5) we see that $yz - xz \in P$. Therefore $xz \leq yz$.

Next we prove $\Longleftarrow$: ie. we need to prove that if $\leq$ satisfies the properties (O1)-(O5) then P satisfies the properties (a) and (b).

Proof of (a): By (O1) either $x \leq 0$ or $x \geq 0$. And by (O2) if $x \neq 0$ then exactly one of the following hold : either $x \leq 0$ or $x \geq 0$. If $x \leq 0$ and $x \neq 0$ then $(0 - x) = -x \in P$. If $x \geq 0$ and $x \neq 0$ then $(x - 0) = x \in P$. Hence (a) holds.

Proof of (b): First part: let $x \in P$ and $y \in P$. Then $0 \leq x$ and $0 \leq y$. Then by P3) and (O4) we have that

$$0 \leq y = 0 + y \leq x + y$$

that is $0 \leq x + y$ which means that $x + y - 0 = x + y \in P$ as desired.

Second part: since x, y are in P, we have that $0 \leq x$ and $0 \leq y$; and we also have that $x \neq 0$ and $y \neq 0$ by (a) which we have independently proved already. By (O5) then $0y \leq xy$. And by homewrok problem 4 we have then that $0 \leq xy$ since $0 = 0y$. Therefore $xy \in P$ ($x \neq 0, y \neq 0$ so $xy \neq 0$).