Math 534H. The Weak Maximum Principle

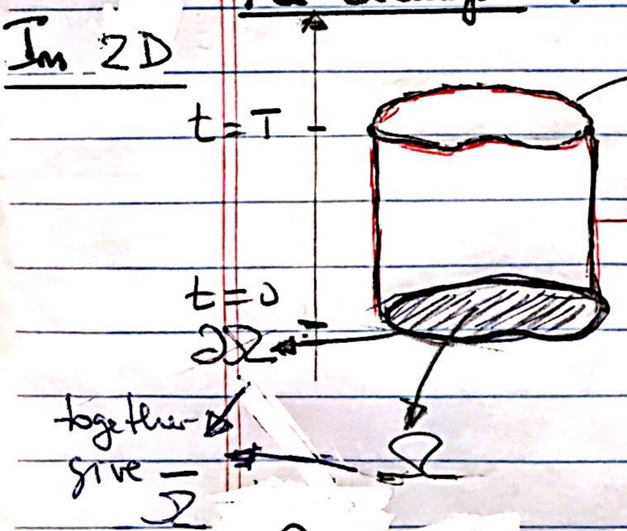
(Salsa's book)

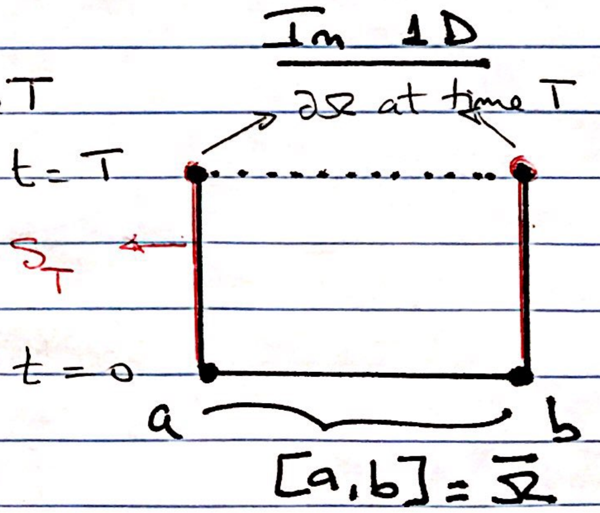
Let $\Omega \subset \mathbb{R}^d$ (open domain), $Q_T = (0,T) \times \Omega$ is a space time cylinder and its parabolic boundary is $\partial Q_T := \left(\overline{\Omega} \times \{t=0\} \right) \cup S_T$

where $S_T := \partial\Omega \times (0,T]$

(here $\partial\Omega$ means the boundary of the domain Ω and $\overline{\Omega}$ means the closure of Ω. Here you may think as $\overline{\Omega}$ being Ω union its boundary: $\overline{\Omega} = \Omega \cup \partial\Omega$)

For example:





Consider a function $f = f(x,t) \leq 0$ $\quad x \in \Omega$, $\; 0<t<T$

and suppose that $u \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ is a solution to the (possible inhomogeneous) heat/diffusion equation in $\Omega \times (0,T)$:

(MAX) says: $\max_{\overline{Q}_T} u = \max_{\partial Q_T} u$

$$u_t - k\Delta u = f \quad (\text{recall } f \leq 0)$$
$$(k>0).$$

Then: $u(x,t)$ obtains its MAXIMUM in the region $\overline{Q}_T$ on ∂Q_T (That is at the bottom or lateral sides)

Similarly suppose that $\tilde{u} \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ is a solution to the (possible inhomogeneous) heat/diffusion equation in $\Omega \times (0,T)$

(MIN) says: $\min_{\overline{Q}_T} \tilde{u} = \min_{\partial Q_T} \tilde{u}$

$$\tilde{u}_t - k\Delta\tilde{u} = \tilde{f} \quad (k>0)$$

where now $\tilde{f} = \tilde{f}(x,t) \geq 0$ on $\Omega \times (0,T)$

Then $\tilde{u}(x,t)$ obtains its MINIMUM in the region $\overline{Q}_T$ on ∂Q_T

Remark: Suppose that $f \equiv 0$ in (MAX) and $\tilde{f} \equiv 0$ in (MIN) and suppose that

$u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ is a solution

to $\boxed{u_t - k\Delta u = 0}$ in $\Omega \times (0,T)$

$(k>0)$.

Then; one has both in this case:

$$\boxed{\begin{aligned} \max_{\overline{Q_T}} u(x,t) &= \max_{\partial Q_T} u(x,t) \\ \text{and} \quad \min_{\overline{Q_T}} u(x,t) &= \min_{\partial Q_T} u(x,t) \end{aligned}}$$

NOTE: In the above recall that $C^{2,1}(Q_T)$ means that u is C^2 in x and C^1 in t for $(x,t) \in Q_T$.

While $C(\overline{Q_T})$ means that $u(x,t)$ is continuous for (x,t) in $\overline{Q_T}$ (including at the boundary).

Since $\overline{Q_T}$ is closed and bounded in $\mathbb{R}^d$, u is in fact uniformly continuous in $\overline{Q_T}$

We will prove (MAX). Once we have (MAX) the proof of (MIN) follows by applying the (MAX) maximum principle to $-u(x,t)$ $(= \tilde{u}(x,t))$ with $-f(x,t)$ $(= \hat{f}(x,t))$.

<u>Proof (MAX)</u>: For simplicity here in this course we consider only the proof in 1D.
(The proof in higher D is similar but a bit more involved).

Let $\varepsilon > 0$ and let $w := u - \varepsilon t$

We wish to obtain the information we seek about the max of u by studying $w(x,t)$ and then letting $\varepsilon \to 0^+$.

Note that on $\overline{Q_T}$ we have $\begin{cases} w \leq u \\ u \leq w + \varepsilon T \end{cases}$

and that on Q_T we have:

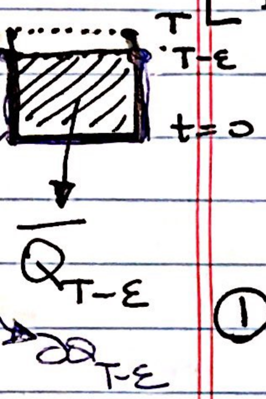
$$(\dagger) \qquad w_t - k w_{xx} = u_t - \varepsilon - k u_{xx}$$

$$= \overbrace{u_t - k u_{xx}} - \varepsilon$$

$$= f - \varepsilon < 0$$

(strictly negative)

CLAIM We would like to prove next that the MAX of w on $\overline{Q}_{T-\varepsilon}$ occurs on $\partial Q_{T-\varepsilon}$:



Suppose that $w(x,t)$ has its maximum at $(x_0, t_0) \in \overline{Q}_{T-\varepsilon}$.

① • And suppose that $0 < t_0 \leq T-\varepsilon$; since if $t_0 = 0$ the claim is obviously true. (since $t_0 = 0$ is part of $\partial Q_{T-\varepsilon}$)

Under this assumption, we have that

$$\begin{cases} w(x,t) < u(x,t) \\ u(x,t) < w(x,t) + \varepsilon T \end{cases}$$

② • Suppose also that $x_0 \in \Omega$, since if $x_0 \in \partial\Omega$, then $(x_0, t_0) \in \partial Q_{T-\varepsilon}$ and the claim would be obviously true.

We would like to draw a contradiction from assuming ① and ② since that contradiction would <u>imply</u> the <u>CLAIM</u> above

Under 1 and 2 :

From calculus $W_x(x_0,t_0) = 0$; and

1 • $W_t(x_0,t_0) = 0$ if $t_0 < T-\varepsilon$

2 • $W_t(x_0,t_0) \geq 0$ if $t_0 = T-\varepsilon$:

Indeed: $\underset{(\text{if } t_0 = T-\varepsilon)}{W_t(x_0,t_0)} = \underbrace{\lim_{\delta \to 0^+} \frac{W(x_0,t_0) - W(x_0,t_0-\delta)}{\delta}}_{\geq 0 \text{ since } \delta > 0 \text{ and also } W(x_0,t_0) \geq W(x_0,t_0-\delta)}$

(where MAX is)

• Since $W(x_0,t_0)$ is the MAXIMUM Value, we can apply Taylor's remainder theorem in x to obtain that for x near x_0 we have:

$$(*) \quad 0 \geq W(x,t_0) - \overset{\text{MAX}}{W(x_0,t_0)} =$$

$$\underbrace{W_x\Big|_{(x_0,t_0)} \cdot (x-x_0)}_{=0} + W_{xx}\Big|_{(x^*,t_0)} \cdot \overset{\text{positive}}{(x-x_0)^2}$$

Where x^* is some point between x and x_0.

Therefore we must have that $W_{xx}(x^*, t_0) \leq 0$ (from ⊛) and by taking the limit as $x \to x_0$ it follows that $x^* \to x_0$ and then

3 • $W_{xx}(x_0, t_0) \leq 0$ (we are using that W_{xx} is continuous in x)

But then we have all in all (from 1 2 3) in all cases

$$W_t(x_0, t_0) - k W_{xx}(x_0, t_0) \geq 0$$

which contradicts (†) in page 4.

Hence the <u>CLAIM</u> at the top of page 5 follows.

Now using $W(x,t) \leq u(x,t)$ and the fact that $\partial Q_{T-\varepsilon} \subset \partial Q_T$ we have thus shown that:

(††) $$\max_{\overline{Q}_{T-\varepsilon}} W = \max_{\partial Q_{T-\varepsilon}} W \leq \max_{\partial Q_{T-\varepsilon}} u \leq \max_{\partial Q_T} u$$

Using (††) above and that $u \leq w + \varepsilon T$ we also have :

(†††) $$\max_{\overline{Q}_{T-\varepsilon}} u \leq \max_{\overline{Q}_{T-\varepsilon}} w + \varepsilon T \leq \varepsilon T + \max_{\partial Q_T} u$$

Now since u is uniformly continuous on $\overline{Q_T}$ we have that

$$\max_{\overline{Q}_{T-\varepsilon}} u \nearrow \max_{\overline{Q}_T} u \quad \text{as } \varepsilon \to 0^+$$

(converges in an increasing fashion).

Thus allowing $\varepsilon \to 0^+$ in (†††) we deduce that :

$$\max_{\overline{Q}_T} u = \lim_{\varepsilon \to 0^+} \max_{\overline{Q}_{T-\varepsilon}} u \leq \lim_{\varepsilon \to 0^+} \left(\varepsilon T + \max_{\partial Q_T} u\right)$$

$$= \max_{\partial Q_T} u$$

$$\leq \max_{\overline{Q}_T} u \implies$$

This means they must all be equal in this chain

$$\max_{\overline{Q}_T} u = \max_{\partial Q_T} u$$

as we wanted to show. #.

We record here a Corollary :

Corollary (Comparison Principle and Stability)

Suppose that v and u are solutions to $v_t - k v_{xx} = f$ and

$$u_t - k u_{xx} = g \quad \text{respectively}$$

(no assumptions on the signs of f or g).

Then: (i) (Comparison Principle): If $v \geq u$ on ∂Q_T and $f \geq g$ then

$$v \geq u \text{ on all of } Q_T$$

(ii) (Stability):

$$\max_{\overline{Q}_T} |v - u| \leq \max_{\partial Q_T} |v - \overset{u}{\cancel{w}}| + T \max_{\overline{Q}_T} |f - g| .$$