1 Waves on the half-line

1.1 Dirichlet boundary condition

We will use the reflection method to solve the boundary value problems associated with the wave equation on the half-line $0<x<\infty$. Let us start with the Dirichlet boundary condition first, and consider the initial boundary value problem

$$\begin{cases} v_{tt}-c^2v_{xx}=0, & 0<x<\infty, 0<t<\infty, \\ v(x,0)=\phi(x), \quad v_t(x,0)=\psi(x), & x>0, \\ v(0,t)=0, & t>0. \end{cases} \tag{1.1}$$

For the vibrating string, the boundary condition of (1.1) means that the end of the string at $x=0$ is held fixed. We reduce the Dirichlet problem (1.1) to the whole line by the reflection method. The idea is again to extend the initial data, in this case ϕ,ψ, to the whole line, so that the boundary condition is automatically satisfied for the solutions of the IVP on the whole line with the extended initial data. Since the boundary condition is in the Dirichlet form, one must take the odd extensions

$$\phi_{\text{odd}}(x)=\begin{cases} \phi(x) & \text{for } x>0, \\ 0 & \text{for } x=0, \\ -\phi(-x) & \text{for } x<0. \end{cases} \qquad \psi_{\text{odd}}(x)=\begin{cases} \psi(x) & \text{for } x>0, \\ 0 & \text{for } x=0, \\ -\psi(-x) & \text{for } x<0. \end{cases} \tag{1.2}$$

Consider the IVP on the whole line with the extended initial data

$$\begin{cases} u_{tt}-c^2u_{xx}=0, & -\infty<x<\infty, 0<t<\infty, \\ u(x,0)=\phi_{\text{odd}}(x), u_t(x,0)=\psi_{\text{odd}}(x). \end{cases} \tag{1.3}$$

Since the initial data of the above IVP are odd, we know from a homework problem that the solution of the IVP, $u(x,t)$, will also be odd in the x variable, and hence $u(0,t)=0$ for all $t>0$. Then defining the restriction of $u(x,t)$ to the positive half-line $x\geq 0$,

$$v(x,t)=u(x,t)\big|_{x\geq 0}, \tag{1.4}$$

we automatically have that $v(0,t)=u(0,t)=0$. So the boundary condition of the Dirichlet problem (1.1) is satisfied for v. Obviously the initial conditions are satisfied as well, since the restrictions of $\phi_{\text{odd}}(x)$ and $\psi_{\text{odd}}(x)$

to the positive half-line are $\phi(x)$ and $\psi(x)$ respectively. Finally, $v(x,t)$ solves the wave equation for $x>0$, since $u(x,t)$ satisfies the wave equation for all $x\in\mathbb{R}$, and in particular for $x>0$. Thus, $v(x,t)$ defined by (1.4) is a solution of the Dirichlet problem (1.1). It is clear that the solution must be unique, since the odd extension of the solution will solve IVP (1.3), and therefore must be unique.

Using d'Alambert's formula for the solution of (1.3), and taking the restriction (1.4), we have that for $x\geq 0$,

$$v(x,t)=\frac{1}{2}[\phi_{\text{odd}}(x+ct)+\phi_{\text{odd}}(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi_{\text{odd}}(s)ds. \tag{1.5}$$

Notice that if $x\geq 0$ and $t>0$, then $x+ct>0$, and $\phi_{\text{odd}}(x+ct)=\phi(x+ct)$. If in addition $x-ct>0$, then $\phi_{\text{odd}}(x-ct)=\phi(x-ct)$, and over the interval $s\in[x-ct,x+ct]$, $\psi_{odd}(s)=\psi(s)$. Thus, for $x>ct$, we have

$$v(x,t)=\frac{1}{2}[\phi(x+ct)+\phi(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds, \tag{1.6}$$

which is exactly d'Alambert's formula.

For $0<x<ct$, the argument $x-ct<0$, and using (1.2) we can rewrite the solution (1.5) as

$$v(x,t)=\frac{1}{2}[\phi(x+ct)-\phi(ct-x)]+\frac{1}{2c}\left[\int_{x-ct}^{0}-\psi(-s)ds+\int_{0}^{x+ct}\psi(s)ds\right].$$

Making the change of variables $s\mapsto -s$ in the first integral on the right, we get

$$\begin{aligned}v(x,t)&=\frac{1}{2}[\phi(x+ct)-\phi(ct-x)]+\frac{1}{2c}\left[\int_{ct-x}^{0}\psi(s)ds+\int_{0}^{x+ct}\psi(s)ds\right]\\&=\frac{1}{2}[\phi(x+ct)-\phi(ct-x)]+\frac{1}{2c}\int_{ct-x}^{x+ct}\psi(s)ds.\end{aligned}$$

One could also use the fact that the integral of the odd function $\psi_{\text{odd}}(s)$ over the symmetric interval $[x-ct,ct-x]$ is zero, thus $\int_{x-ct}^{x+ct}\psi_{\text{odd}}(s)ds=\int_{ct-x}^{x+ct}\psi(s)ds$.

The two different cases giving different expressions are illustrated in Figures 1.1 and 1.2 below. Notice how one of the characteristics from a point with $x_0<ct_0$ gets reflected from the "wall" at $x=0$ in Figure 1.2.

Combining the two expressions for $v(t,x)$ over the two regions, we arrive at the solution

$$v(x,t)=\begin{cases}\frac{1}{2}[\phi(x+ct)+\phi(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds, & \text{for } x>ct\\ \frac{1}{2}[\phi(x+ct)-\phi(ct-x)]+\frac{1}{2c}\int_{ct-x}^{x+ct}\psi(s)ds, & \text{for } 0<x<ct.\end{cases} \tag{1.7}$$

The minus sign in front of $\phi(ct-x)$ in the second expression above, as well as the reduction of the integral of ψ to the smaller interval are due to the cancellation stemming from the *reflected wave*. The next example

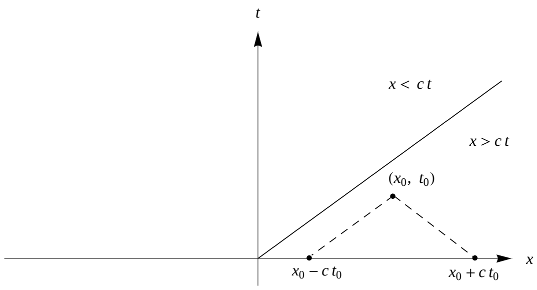

Figure 1.1: The case with $x_0 > ct_0$.

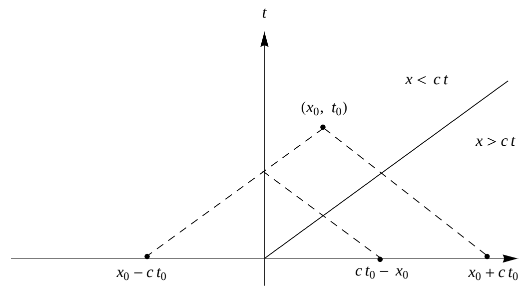

Figure 1.2: The case with $x_0 < ct_0$.

illustrates this phenomenon.

1.2 Neumann boundary condition

For the Neumann problem on the half-line,

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0, \quad 0 < x < \infty, 0 < t < \infty, \\ w(x,0) = \phi(x), \quad w_t(x,0) = \psi(x), \quad x > 0, \\ w_x(0,t) = 0, \quad t > 0, \end{cases} \tag{1.8}$$

we use the reflection method with even extensions to reduce the problem to an IVP on the whole line. Define the even extensions of the initial data

$$\phi_{\text{even}} = \begin{cases} \phi(x) & \text{for } x \geq 0, \\ \phi(-x) & \text{for } x \leq 0, \end{cases} \qquad \psi_{\text{even}} = \begin{cases} \psi(x) & \text{for } x \geq 0, \\ \psi(-x) & \text{for } x \leq 0. \end{cases} \tag{1.9}$$

and consider the following IVP on the whole line

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \quad -\infty < x < \infty, 0 < t < \infty, \\ u(x,0) = \phi_{\text{even}}(x), \quad u_t(x,0) = \psi_{\text{even}}(x). \end{cases} \tag{1.10}$$

Clearly, the solution $u(x,t)$ to the IVP (1.10) will be even in x, and since the derivative of an even function is odd, $u_x(x,t)$ will be odd in x, and hence $u_x(0,t) = 0$ for all $t > 0$. Similar to the case of the Dirichlet problem, the restriction

$$w(x,t) = u(x,t)\big|_{x \geq 0}$$

will be the unique solution of the Neumann problem (1.8).

Using d'Alambert's formula for the solution $u(x,t)$ of (1.10), and taking the restriction to $x \geq 0$, we get

$$w(x,t) = \frac{1}{2}[\phi_{\text{even}}(x+ct) + \phi_{\text{even}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{even}}(s)ds. \tag{1.11}$$

One again needs to consider the two cases $x > ct$ and $0 < x < ct$ separately. Notice that with the even extensions we will get additions, rather than cancellations. Using the definitions (1.9), the solution (1.11) can be written as

$$w(x,t) = \begin{cases} \frac{1}{2}[\phi(x+ct)+\phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds, & \text{for } x > ct \\ \frac{1}{2}[\phi(x+ct)+\phi(ct-x)] & \\ \qquad + \frac{1}{2c}\left[\int_0^{ct-x} \psi(s)ds + \int_0^{x+ct} \psi(s)ds\right], & \text{for } 0 < x < ct. \end{cases}$$

The Neumann boundary condition corresponds to a vibrating string with a free end at $x = 0$, since the string tension, which is proportional to the derivative $v_x(x,t)$, vanishes at $x = 0$. In this case the reflected wave adds to the original wave, rather than canceling it.

1.3 Conclusion

We derived the solution to the wave equation on the half-line. That is, we reduced the initial/boundary value problem to the initial value problem over the whole line through appropriate extension of the initial data. In this case the characteristics nicely illustrate the reflection phenomenon. We saw that the characteristics that hit the initial data after reflection from the boundary wall $x = 0$ carry the values of the initial data with a minus sign in the case of the Dirichlet boundary conditions, and with a plus sign in the case of the Neumann boundary conditions. This corresponds to our intuition of reflected waves from a fixed end, and free end respectively.