Chapter 7. The Hille-Yosida Theorem

$\mathcal{H}$ = Hilbert

Theorem 7.4 (Hille-Yosida): Let A be a maximal monotone operator. That is:

- A is unbounded linear $A : D(A) \subset \mathcal{H} \to \mathcal{H}$
  $(Av, v) \geq 0 \quad \forall v \in D(A)$
- $R(I + A) = \mathcal{H}$: $\forall f \in \mathcal{H} \; \exists \; u \in D(A) /$
  $u + Au = f$

Then given $u_0 \in D(A)$ there exists a unique function $u \in C^1([0,+\infty); \mathcal{H}) \cap C([0,+\infty); D(A))$

satisfying:

$$(\dagger) \quad \begin{cases} \dfrac{du}{dt} + Au = 0 \\ u(0) = u_0 \end{cases}$$

Moreover:
$$|u(t)| \leq |u_0| \quad \forall t \geq 0$$

$$\left|\frac{du}{dt}(t)\right| = |Au(t)| \leq |Au_0| \quad \forall t \geq 0$$

Remark 1: $\mathcal{D}(A)$ is equipped with the graph norm $|v|+|Av|$ or equivalently with the Hilbert norm $\left(|v|^2+|Av|^2\right)^{1/2}$.

Remark 2: Point of Theorem is to reduce the study of evolution eq. (†) to the stationary equation $u+Au=f$; assuming one knows that A is monotone.

Proof of Theorem 7.4

(I) Uniqueness: Let u and $\tilde{u}$ be two solutions to (†) then if we let $w = u-\tilde{u}$ we have by linearity $\qquad w(0)=0$

$$0 = \left(\frac{dw}{dt} + Aw, w\right)$$

$$= \left(\frac{dw}{dt}, w\right) + (Aw, w)$$

$$\Rightarrow \underbrace{\left(\frac{dw}{dt}, w\right)}_{\substack{\| \\ \frac{1}{2}\frac{d}{dt}|w|^2}} = -\underbrace{(Aw, w)}_{\geq 0 \text{ by monotone.}} \leq 0$$

using $w \in C^1([0,\infty); \mathcal{H})$

$\Rightarrow t \longmapsto M(w)(t)$ is nonincreasing

where $M(w)(t) = \frac{1}{2}|w|^2(t)$. $\forall t \geq 0$

But $w(0) = 0 \Rightarrow |w|^2(0) = 0$

and $M(w)(t) \leq M(w)(0) = 0$

$$\Rightarrow M(w)(t) \equiv 0 \quad \forall t \geq 0$$

$$\Rightarrow |w|(t) \equiv 0 \quad \forall t \geq 0.$$

$$\Rightarrow x(t) = \tilde{x}(t) \quad \forall t \geq 0.$$

\#.

(II) Existence : Idea is to approximate

($\lambda > 0$) A by $A_\lambda = \frac{1}{\lambda}(I - J_\lambda)$ where $J_\lambda = (I + \lambda A)^{-1}$

the Yosida approximation/regularization of A.

(Recall $\|J_\lambda\|_{\mathcal{L}(\mathcal{H})} \leq 1$ J_λ = resolvent of A).

And solve (+) with A_λ using Th. 7.3 (Picard) and obtain estimates which are independent of λ so that then we can pass to the limit $\lambda \to 0^+$ and obtain a solution for (+) with A.

So consider

$$(\dagger)_\lambda \begin{cases} \dfrac{du_\lambda}{dt} + A_\lambda u_\lambda = 0 \quad \text{on } [0,\infty). \\ u_\lambda(0) = u_0 \in \underline{\underline{D(A)}} \end{cases}$$

The proof is now divided into 5 steps a) - e).

II a) We prove an auxiliary Lemma:

Lemma 7.1: Let $w \in C^1([0,\infty); \mathcal{H})$ be a function / $\dfrac{dw}{dt} + A_\lambda w = 0$ on $[0,+\infty)$

Then the functions $t \longmapsto |w(t)|$ and

$$t \longmapsto |A_\lambda w(t)|$$

are nonincreasing on $[0,+\infty)$.

Proof Lemma 7.1: $\left(\dfrac{dw}{dt}, w\right) + (A_\lambda w, w) = 0$ (*)

and by Prop 7.2e) $(A_\lambda w, w) \geq 0 \Rightarrow$ (as before)

$\dfrac{1}{2}\dfrac{d}{dt}|w|^2 \leq 0 \Rightarrow |w(t)|$ is nonincreasing

On the other hand, since A_λ is linear bdd one can

by induction deduce from (*) that $w \in C^\infty([0,\infty); \mathcal{H})$ and also

that $\dfrac{d}{dt}\left(\dfrac{dw}{dt}\right) + A_\lambda\left(\dfrac{dw}{dt}\right) = 0$ [by induction

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(same as before)

Now, we repeat with $\frac{dw}{dt}$ to get that $\left|\frac{dw}{dt}(t)\right|$ is nonincreasing. In fact for any order k, $\left|\frac{d^k w}{dt^k}(t)\right|$ is nonincreasing #

- Now from this Lemma + the fact $|A_\lambda u_0| \le |Au_0|$ we have that

$\mathbb{D}^T$

$\measuredangle$

$\|$

$\measuredangle^T$

(**)
$$\begin{cases} |u_\lambda(t)| \le |u_0| & \forall t \ge 0, \lambda > 0 \\ \left|\frac{du_\lambda}{dt}(t)\right| = |A_\lambda u_\lambda(t)| \le |Au_0| & \forall t \ge 0, \lambda > 0 \end{cases}$$

II b) Now we want to show that $u_\lambda(t)$ converges as $\lambda \to 0$. We'll denote this limit $u(t)$. We'll also show the convergence is uniform on every bounded interval $[0,T]$, $T > 0$.

(Don't know this is a solution yet)

For every $\lambda, \mu > 0$ we have

$$\frac{du_\lambda}{dt} + A_\lambda u_\lambda - \frac{du_\mu}{dt} - A_\mu u_\mu = 0$$

and thus

(†) $$0 = \frac{1}{2}\frac{d}{dt}|u_\lambda(t) - u_\mu(t)|^2 + (A_\lambda u_\lambda(t) - A_\mu u_\mu(t), u_\lambda(t) - u_\mu(t))$$

Dropping t for simplicity we write

$$(A_\lambda u_\lambda - A_\mu u_\mu, u_\lambda - u_\mu) =$$

$$= (A_\lambda u_\lambda - A_\mu u_\mu, u_\lambda - J_\lambda u_\lambda + J_\lambda u_\lambda - J_\mu u_\mu + J_\mu u_\mu - u_\mu)$$

$I - J_\lambda = \lambda A_\lambda$

$$= (A_\lambda u_\lambda - A_\mu u_\mu, \lambda A_\lambda u_\lambda - \mu A_\mu u_\mu) +$$

$\oplus\oplus$ $$+ (A(J_\lambda u_\lambda - J_\mu u_\mu), J_\lambda u_\lambda - J_\mu u_\mu)$$

by monotonicity

J_λ bdd

$$\geq (A_\lambda u_\lambda - A_\mu u_\mu, \lambda A_\lambda u_\lambda - \mu A_\mu u_\mu)$$

It follows from $\oplus\oplus$ $\oplus$ and $\oplus\oplus$ that

$$\boxed{\frac{1}{2}\frac{d}{dt}|u_\lambda - u_\mu|^2 \leq 2(\lambda+\mu)|Au_0|^2}$$

$\Rightarrow$

Integrate in t b/t 0 and t $$|u_\lambda(t) - u_\mu(t)|^2 \leq 4(\lambda+\mu)t|Au_0|^2$$

$\oplus\oplus\oplus$ $\Rightarrow$ $$|u_\lambda(t) - u_\mu(t)| \leq 2\sqrt{(\lambda+\mu)t}\,|Au_0|$$

$\Rightarrow$ $\{u_\lambda(t)\}_\lambda$ is a Cauchy seq. for each fixed $t \geq 0$.

Call limit $u(t)$. Now pass to the limit in $\oplus\oplus\oplus$ as $\mu \to 0$ to get

$t \in [0, T]$

$$|u_\lambda(t) - u(t)| \leq 2\sqrt{\lambda t}\,|Au_0| \leq 2\sqrt{\lambda T}\,|Au_0|$$

Therefore the convergence is uniform in $t \in [0, T]$, $\forall T > 0 \Rightarrow$

$$u \in C([0, +\infty); H).$$

III c) Assume now $\boxed{u_0 \in D(A^2)}$

(ie. $u_0 \in D(A)$ and $Au_0 \in D(A)$)

we WTS $\frac{du_\lambda}{dt}(t)$ converges as $\lambda \to 0$ to some limit and that convergence is uniform on every bounded interval $[0, T]$.

Set $v_\lambda = \frac{du_\lambda}{dt} \Rightarrow \frac{dv_\lambda}{dt} + A_\lambda v_\lambda = 0$

By similar argument as II b) and Lemma 7.1 we can show

$$\frac{1}{2}\frac{d}{dt}|v_\lambda - v_\mu|^2 \leq 2(\lambda + \mu)\, |A^2 u_0|^2$$

$\Rightarrow$ $v_\lambda(t) = \frac{du_\lambda(t)}{dt}$ converges as $\lambda \to 0$
as before
to some limit and that the convergence is uniform on every bounded interval $[0, T]$.

III d) Here assuming that $\boxed{u_0 \in D(A^2)}$ we prove here that u is a solution of $(+)$

From II b) and II c) we know that for all $T < \infty$

$$\begin{cases} u_\lambda(t) \longrightarrow u(t) \quad \lambda \to 0 \text{ uniformly on } [0,T] \\ \dfrac{du_\lambda}{dt}(t) \text{ converges as } \lambda \to 0 \text{ uniformly on } [0,T]. \end{cases}$$

HMWK: Spell this OUT.

Hence $u \in C^1([0,+\infty), \mathcal{H})$ and that

$$\frac{du_\lambda}{dt}(t) \longrightarrow \frac{du}{dt}(t) \text{ as } \lambda \to 0, \text{ uniformly on } [0,T].$$

(++) Rewrite $(+)_\lambda$ as $\dfrac{du_\lambda}{dt}(t) + \overbrace{A(J_\lambda u_\lambda)}^{A_\lambda}(t) = 0$

Note that $J_\lambda u_\lambda(t) \longrightarrow u(t)$ as $\lambda \to 0$

since $|J_\lambda u_\lambda(t) - u(t)| \leq$

$$|J_\lambda u_\lambda(t) - J_\lambda u(t)| + |J_\lambda u(t) - u(t)|$$

$$\leq |u_\lambda(t) - u(t)| + |J_\lambda u(t) - u(t)| \xrightarrow[\lambda \to 0]{} 0$$

Apply now that A has a closed graph to deduce from (††) that $u(t) \in D(A)$ $\forall t \geq 0$ and that $\frac{du}{dt}(t) + Au(t) = 0$

Since $u \in C^1([0,+\infty), \mathcal{H})$, $t \longmapsto Au(t)$ is continuous from $[0,+\infty) \longrightarrow \mathcal{H}$ and thus $u \in C([0,+\infty); D(A))$. All in all we have a solution u to (†) /

$$|u(t)| \leq |u_0| \quad \forall t \geq 0 \quad \text{and}$$

$$\left|\frac{du}{dt}(t)\right| = |Au(t)| \leq |Au_0| \quad \forall t \geq 0.$$

We need to resolve the extra assumption $u_0 \in D(A^2)$

III e) We now finish the proof of the main theorem. Let's first prove the following

Lemma 7.2: Let $u_0 \in D(A)$. Then $\forall \varepsilon > 0$ $\exists\, \tilde{u}_0 \in D(A^2)$ / $|u_0 - \tilde{u}_0| < \varepsilon$ and $|Au_0 - A\tilde{u}_0| < \varepsilon$. In other words, $D(A^2)$ is dense in $D(A)$ w.r.t. graph norm.

Proof (Lemma 7.2): set $\tilde{u}_0 = J_\lambda u_0$ for some appropriate λ TBD. $\tilde{u}_0 \in D(A)$ and $\tilde{u}_0 + \lambda A\tilde{u}_0 = u_0$

Then, $A\tilde{u}_0 \in D(A)$ ie. $\tilde{u}_0 \in D(A^2)$.

~~On the other hand by PROP 7.2 we know that~~

$$\lim_{\lambda \to 0} |J_\lambda u_0 - u_0| = 0$$

A_λ

$$\lim_{\lambda \to 0} |J_\lambda A u_0 - A u_0| = 0 \quad \text{and} \quad J_\lambda A u_0 = A J_\lambda u_0$$

Hence we get the conclusion by choosing $\lambda > 0$ sufficiently small. #.

Now we finish the proof of Theorem 7.4. :

Given $u_0 \in D(A)$, we use Lemma 7.2 to construct a sequence $u_{0,n} \in D(A^2)$ / $u_{0,n} \longrightarrow u_0$ and $Au_{0,n} \longrightarrow Au_0$. By II d) we know $\exists$ a solution u_n to

$$(NE) \begin{cases} \dfrac{du_n}{dt} + Au_n = 0 & \text{on } [0, +\infty) \\ u_n(0) = u_{0,n}. \end{cases}$$

We have, $\forall t \geq 0$ $|u_n(t) - u_m(t)| \leq |u_{0,n} - u_{0,m}| \xrightarrow[m,n\to\infty]{} 0$

and

$$\left|\frac{du_n}{dt}(t) - \frac{du_m}{dt}(t)\right| \leq |Au_{0,n} - Au_{0,m}| \xrightarrow[n,m\to\infty]{} 0$$

HMWK: Spell out

Therefore,

$$\left.\begin{aligned} u_n(t) &\longrightarrow u(t) \text{ uniformly} \\ \frac{du_n}{dt}(t) &\longrightarrow \frac{du}{dt}(t) \text{ uniformly} \end{aligned}\right\} \forall t \geq 0$$

with $u \in C^1([0,+\infty); \mathcal{H})$. Passing to the limit in (NE) (as $n \to \infty$) and using that A is closed operator we see that $u(t) \in D(A)$ and u satisfies (+). From (+) we deduce that $u \in C([0,+\infty); D(A))$.

Note

Remark: Let A be a maximal monotone operator and $\eta \in \mathbb{R}$. The problem

$$\begin{cases} \dfrac{du}{dt} + Au + \eta u = 0 \text{ on } [0,+\infty) \\ u(0) = u_0 \end{cases}$$

reduces to problem (+) by setting $v(t) = e^{\eta t} u(t)$.

Then

$$\begin{cases} \dfrac{dv}{dt} + Av = 0 \quad \forall t \geq 0 \\ v(0) = u_0 \end{cases}$$