FROM CHAPTER VI Section 3 (Reed & Simon Vol I).

Definition: Let X be a Banach space and let $T \in \mathcal{L}(X)$. A complex number λ is said to be in the resolvent $\rho(T)$ if $\lambda I - T$ $(\lambda - T)$ is a bijection with a bounded inverse.

$R_\lambda(T) = (\lambda I - T)^{-1}$ is called the resolvent of T at λ. If $\lambda \notin \rho(T)$ then λ is said to be in the SPECTRUM $\sigma(T)$ of T

Remark: By the inverse mapping theorem, $\lambda I - T$ automatically has a bounded inverse if it is bijective. We distinguish two subsets of the spectrum

Definition: Let $T \in \mathcal{L}(X)$.

a) An $x \neq 0$ which satisfies $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$ is called an eigenvector of T, λ is called the corresponding eigenvalue. If λ is an eigenvalue then $\lambda I - T$ is not injective so λ is in the spectrum of T. The set of all eigenvalues is called the point spectrum

b) If λ is not an eigenvalue and if $R(\lambda I - T)$ is NOT dense then λ is said to be in the residual spectrum

Remark: Residual spectrum doesn't occur for a large class of operators such as self adjoints operators.

Theorem VI.5. (p. 190): Let X is Banach and suppose $T \in \mathcal{L}(X)$. Then $\rho(T)$ is an open subset of $\mathbb{C}$ and $R_\lambda(T)$ is an analytic $\mathcal{L}(X)$-valued function on each component (maximal connected subset). For any two points λ, μ in $\rho(T)$, $R_\lambda(T)$ and $R_\mu(T)$ commute

FIRST RESOLVENT FORMULA

$$R_\lambda(T) - R_\mu(T) = (\mu - \lambda) R_\mu(T) R_\lambda(T)$$

Corollary: Let X be Banach, $T \in \mathcal{L}(X)$. Then the spectrum of T is not empty

(Corollary) Proof: Formally for $|\lambda|$ large,

NEUMANN SERIES FOR $R_\lambda(T)$

$$R_\lambda(T) = \frac{1}{\lambda}\underbrace{\left(I + \sum_{n=1}^{\infty}\left(\frac{T}{\lambda}\right)^n\right)}_{(I - T/\lambda)^{-1}}$$

So if $|\lambda| > \|T\|$ the series on the right converges in norm and it is easily checked that for such λ its limit is the inverse of $(\lambda I - T)$. Thus as $|\lambda| \to \infty$ $\|R_\lambda(T)\| \to 0$. If $\sigma(T)$ were empty $R_\lambda(T)$ would be an entire bounded analytic function. By Liouville's theorem $R_\lambda(T)$ would be zero which is a contradiction $\therefore \sigma(T) \neq \emptyset$ *

• The proof of the corollary also shows that $\sigma(T) \subset \overline{D(0, \|T\|)} \subset \mathbb{C}$.

In fact :

Definition: Let $r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$

$r(T)$ = spectral radius of T

Theorem VI 6. (P. 192) : Let X be Banach $T \in \mathcal{L}(X)$. Then $\lim_{n\to\infty} \|T^n\|^{1/n}$ $\exists$ and $= r(T)$

If X is a Hilbert space and A is self-adjoint then $r(A) = \|A\|$.

IMP.

Theorem VI 7. (Phillips p. 182) Let X be Banach $T \in \mathcal{L}(X)$. Then $\sigma(T) = \sigma(T')$ and $R_\lambda(T') = R_\lambda(T)'$. If $\mathcal{H}$ is a Hilbert space $\sigma(T^*) = \{\lambda / \bar{\lambda} \in \sigma(T)\}$ and $R_{\lambda}(T^*) = R_{\lambda}(T^*)^*$

see pg. 186

T^* on Hilbert is $C^{-1}T'C$ where $C : \mathcal{H} \to \mathcal{H}^*$ is the map $y \longmapsto (y, \cdot)$ in $\mathcal{H}^*$
C is conjugate linear isometry which is surjective by Riesz Representation Lemma (pg. 43).

$T^* : \mathcal{H} \to \mathcal{H}$.
$(T : \mathcal{H} \to \mathcal{H}$ bdd linear$)$
$(x, Ty) = (T^*x, y)$

Proposition (p. 194) : X Banach $T \in \mathcal{L}(X)$. Then

a) If $\lambda \in$ residual sp.$(T) \Rightarrow \lambda \in$ point sp.(T')

b) If $\lambda \in$ point sp$(T) \Rightarrow$ either $\lambda \in$ point sp(T') or $\lambda \in$ residual sp(T')

$\in \mathcal{L}(\mathcal{H})$

Theorem VI 8 (p. 194): Let T be selfadjoint on $\mathcal{H}$ Hilbert. Then

a) T has no residual spectrum

b) $\sigma(T)$ is a subset of $\mathbb{R}$ } IMP

HMWK → c) Eigenvectors corresponding to distinct eigenvalues of T are orthogonal.

Proof: If λ, μ are real we compute

$$\|[T-(\lambda+i\mu)]x\|^2 = \|(T-\lambda)x\|^2 + \mu^2\|x\|^2$$

$(x \in \mathcal{H})$.

$$\geq |\mu|^2\|x\|^2 \qquad (\dagger)$$

So if $\mu \neq 0$ $(T-(\lambda+i\mu))$ is a 1-1 operator and has bounded inverse on its range,

(WHY?) which is closed. If $\mathrm{Ran}(T-(\lambda+i\mu)) \neq \mathcal{H}$ then (previous Proposition) $\lambda - i\mu$ would be in the point spectrum of T, contradicting $(\dagger)$.

Hence, if $\mu \neq 0$ $\lambda + i\mu \in \rho(T)$. $\Rightarrow$ b)

If λ (real) $\in$ residual spectrum of $T \Rightarrow$ ($\bar{\lambda} = \lambda$) would be in the point spectrum of T^* $(=T)$ which is impossible since pt & residual are disjoint

This proves a). Part c) is an exercise (HMWK).

- Reed & Simon then discuss Compact op. & Hilbert-Schmidt (read on your own)

In Chapter VIII unbounded self adjoint operators spectral theorem

Chapter VII: The Spectral Theorem for self-adjoint operators (bounded) A.

Essentially one can phrase it by stating that "every bounded s.a. operator is a 'multiplication operator'".

What does this mean? Given a bounded self adjoint operator on a Hilbert space $\mathcal{H}$ we can always find a measure μ on a measure space M and a unitary operator $U : \mathcal{H} \longrightarrow L^2(M, d\mu)$ so that

μ = spectral measure

$$(UAU^{-1}f)(x) = F(x) \cdot f(x)$$

for some bounded real-valued measurable function F on M.

This is a generalization to infinite dimensions of the fact that a s.a. $n \times n$ matrix can be diagonalized.

In practice M will be a union of copies of $\mathbb{R}$ and F will be x. So the key part of the proof is the construction of certain measures.

We'll do this in Section 2 (Ch VII).

$\boxed{\text{First we will make sense of } f(A) \text{ for } f \text{ continuous function}}$ Then in section 2 we'll study measures defined by functionals

$$f \longmapsto \langle \psi, f(A)\psi \rangle \quad \text{for fixed } \psi \in \mathcal{H}.$$

If $f(x) = \sum_{n=1}^{N} c_n x^n$ is a polynomial we'd want $f(A)$ to be $\sum_{n=1}^{N} c_n A^n$.

If $f(x) = \sum_{n=1}^{\infty} c_n x^n$ is a power series with radius of convergence $\|x\| < R$ then we want $f(A) = \sum_{n=1}^{\infty} c_n A^n$ which converges in $\mathcal{L}(\mathcal{H})$ if $\|A\| < R$. Note that f is real analytic in a domain including all of $\sigma(A)$.

$\boxed{\text{This will be a reasonable hypothesis to properly define } f(A)}$

So far we haven't used A is s.a (just bdd.) But the special property of being s.a (or more generally normal $NN^* = N^*N$; e.g. unitary op. $N^* = N^{-1}$, s.a. operators.) is that $\boxed{\|P(A)\| = \sup_{\lambda \in \sigma(A)} |P(\lambda)|}$ for any polynomial P so one can use the extension theorem for bounded linear transformations (BLT theorem) see Theorem I.7 pg. 9) to extend the functional calculus to continuous functions.

Theorem VII.1: Let A be a s.a. operator on a Hilbert space $\mathcal{H}$. Then there exists a unique map $\phi : C(\sigma(A)) \longrightarrow \mathcal{L}(\mathcal{H})$ with the following properties:

(a) ϕ is an algebraic $*$-homomorphism ie.

$$\begin{cases} \phi(fg) = \phi(f)\phi(g) & \phi(\lambda f) = \lambda\phi(f) \\ \phi(1) = I & \phi(\bar{f}) = \phi(f)^* \end{cases}$$

(b) ϕ is continuous, i.e. $\|\phi(f)\|_{\mathcal{L}(\mathcal{H})} \leq c\|f\|_\infty$

(c) Let $f(x) = x$ then $\phi(f) = A$

Moreover: ϕ has the additional properties

(d) If $A\psi = \lambda\psi \Rightarrow \phi(f)\psi = f(\lambda)\psi$

(e) $\sigma(\phi(f)) = \{f(\lambda) / \lambda \in \sigma(A)\}$

(spectral mapping theorem)

(f) If $f \geq 0 \Rightarrow \phi(f) \geq 0$

(g) $\|\phi(f)\|_{\mathcal{L}(\mathcal{H})} = \|f\|_\infty$ (strengthening b))

(NOTATION: $\phi_A(f)$, $f(A)$ or $\phi(f)$)

Recall: Weierstrass theorem shows the set of polynomials is dense in continuous functions

Now a) and c) uniquely determine $\phi(P)$ for any polynomial $P(x)$. Then by Weierstrass theorem set of polynomials is dense in $C(\sigma(A))$ so the key is to show

$$\|P(A)\|_{\mathcal{L}(\mathcal{H})} = \|P(x)\|_{C(\sigma(A))} \equiv \sup_{\lambda\in\sigma(A)} |P(\lambda)|$$

$\exists!$ of ϕ then follows from the BLT theorem.

To prove the crucial equality, we first ~~show a special case of e) which holds~~ for arbitrary bounded operators.

(pg. 223 Reed-Simon) **Lemma 1**: Let $P(x) = \sum_{n=0}^{N} a_n x^n$. Let $P(A) = \sum_{n=0}^{N} a_n A^n$. Then

$$\sigma(A) = \{ P(\lambda) \,/\, \lambda \in \sigma(A) \}.$$

Then

Lemma 2: Let A be a bounded s.a. operator. Then $\|P(A)\| = \sup_{\lambda \in \sigma(A)} |P(\lambda)|$.

Now we are ready for the proof of Theorem VII

Let $P(A) := \phi(P)$. Then $\|\phi(P)\|_{\mathcal{L}(\mathcal{H})} = \|P\|_{C(\sigma(A))}$ so ϕ has a unique extension to the closure of the polynomials in $C(\sigma(A))$. ~~Since the polynomials are an algebra~~ containing 1, conjugates and separating

(HMWK) points, ~~this closure is all of~~ $C(\sigma(A))$

Properties a) b) c) g) are immediate and if $\tilde{\phi}$ obeys a) b) c) $\Rightarrow$ it agrees w/ ϕ on polynomials and by continuity on $C(\sigma(A))$. To prove d) one notes that $\phi(P)\psi = P(\lambda)\psi$ and applies continuity from here.

f) follows from the fact that if $f \geq 0 \Rightarrow$ $f = g^2$ with g real in $C(\sigma(A))$. $\Rightarrow$

$$\phi(f) = \phi(g)^2 \quad \phi(g) \text{ self-adjoint}$$

$\Rightarrow$ $\phi(f) \geq 0$.

e) is a HMWK.

<u>Remark:</u> From g) we see that

$$\|(A-\lambda)^{-1}\| = (\text{dist}(\lambda, \sigma(A)))^{-1} \text{ if}$$

A is bdd. s.a. and $\lambda \notin \sigma(A)$

II) <u>The SPECTRAL MEASURE</u>

Let A be bounded, linear, s.a. operator, and let $\psi \in \mathcal{H}$. Then $f \longmapsto (\psi, f(A)\psi)$ is a positive linear functional on $C(\sigma(A))$. Thus, by the Riesz-Markov theorem, $\exists!$ msr μ_ψ on the compact set $\sigma(A)$ with

$$(\psi, f(A)\psi) = \int_{\sigma(A)} f(\lambda)\, d\mu_\psi(\lambda)$$

μ_ψ is called the SPECTRAL MEASURE associated to ψ.

(Th. IV.14) The Riesz-Markov Theorem states: Let X be compact Hausdorff space. For any positive linear functional ℓ on $C(X)$ $\exists!$ Baire msr μ on X with $\ell(f) = \int f\, d\mu$

See [RS] Section IV.4 Measure Theory on Compact Spaces.

Remark: This theorem states that the dual of $C(X)$ can be interpreted as the space of Baire measures. (Baire msrs are finite).

Spectral msr μ_ψ allows us to extend the functional calculus to the set of bounded Borel functions on $\mathbb{R}$ ($\mathcal{B}(\mathbb{R})$). Let $g \in \mathcal{B}(\mathbb{R})$. Then we want to define $g(A)$ /

$$(\psi, g(A)\psi) = \int_{\sigma(A)} g(\lambda)\, d\mu_\psi(\lambda) \qquad (\dagger)$$

The polarization identity:

$$(x,y) = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) - \frac{i}{4}\left(\|x+y\|^2 - \|x-iy\|^2\right)$$

would allow one to recover $(\psi, g(A)\phi)$ from $(\dagger)$ and then the Riesz representation lemma let's us construct $g(A)$:

Theorem VII.2: (Spectral theorem - functional calculus form). Let A be bdd s.a on $\mathcal{H}$. There is a unique map $\tilde{\phi} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ so that

a) $\tilde{\phi}$ is an algebraic $*$-homomorphism.

b) $\tilde{\phi}$ is norm continuous: $\|\tilde{\phi}(f)\|_{\mathcal{L}(\mathcal{H})} \leq \|f\|_\infty$

c) Let $f(x) = x \Rightarrow \tilde{\phi}(f) = A$

d) Suppose $f_n(x) \to f(x)\ \forall x$ and $\{\|f_n\|_\infty\}_{n\geq 1}$ is bdd. Then $\tilde{\phi}(f_n) \to \tilde{\phi}(f)$ strongly (in $\mathcal{L}(\mathcal{H})$)

Moreover, $\tilde{\phi}$ has the properties:

(e) If $A\psi = \lambda\psi \Rightarrow \tilde{\Phi}(f)\psi = f(\lambda)\psi$

(f) If $f \geq 0 \Rightarrow \tilde{\Phi}(f) \geq 0$

(g) If $BA = AB \Rightarrow \tilde{\Phi}(f)B = B\tilde{\Phi}(f)$.

DEFINITION: A vector $\psi \in \mathcal{H}$ is cyclic vector for A if finite linear combinations of elements $\{A^n\psi\}_{n=0}^{\infty}$ are dense in $\mathcal{H}$

s.a. bdd op on $\mathcal{H}$

Not all operators have cyclic vectors. But when they do we have the following

LEMMA 1: Let A be bdd s.a. with cyclic vector ψ. Then $\exists$ a unitary operator

$U : \mathcal{H} \to L^2(\sigma(A), d\mu_\psi)$ with

spectral msr ass. to ψ (cyclic)

$$(UAU^{-1}f)(\lambda) = \lambda f(\lambda).$$

in L^2-sense.

Proof: Let us define $U\phi(f)\psi \equiv f$ where f is continuous. U is essentially the inverse of the map ϕ of Theorem VII.1.

To show U is well defined we compute

$$\|\phi(f)\psi\|^2 = (\psi, \phi^*(f)\phi(f)\psi)$$
$$= (\psi, \phi(\bar{f} f)\psi)$$
$$= \int |f(\lambda)|^2 \, d\mu_\psi$$

Therefore, if $f = g$ a.e. w.r.t. μ_ψ then $\phi(f)\psi = \phi(g)\psi$.

Then U is well-defined on $\{\phi(f)\psi \,/\, f \in C(\sigma(A))\}$ and is norm preserving.

Since ψ is cyclic $\underbrace{\overline{\{\phi(f)\psi \,/\, f \in C(\sigma(A))\}}}_{= \mathcal{H}}$

so by the BLT theorem,

U extends to an isometric map of $\mathcal{H}$ into $L^2(\sigma(A), d\mu_\psi)$. Since $C(\sigma(A))$ is dense in L^2, $\operatorname{Ran} U = L^2(\sigma(A), d\mu_\psi)$

Finally, if $f \in C(\sigma(A))$, then

$$(UAU^{-1}f)(\lambda) = [UA\phi(f)](\lambda)$$
$$=$$

$$= \left[U \phi(x f) \right](\lambda)$$

$$= \lambda f(\lambda) .$$

By continuity, this extends from $f \in C(\sigma(A))$ to $f \in L^2$. #

To extend the lemma to arbitrary A we need to know that A has a family of invariant subspaces spanning $\mathcal{H}$ so that A is cyclic on each subspace.

<u>LEMMA 2</u>: Let A be s.a. on a separable $\mathcal{H}$ (Hilbert space). Then there is a direct sum decomposition $\mathcal{H} = \bigoplus_{n=1}^{N} \mathcal{H}_n$ with $N = 1, 2, \ldots$ or ∞ so that

a) A leaves each $\mathcal{H}_n$ <u>invariant</u>; <u>ie</u>.

s.a op.
$$\psi \in \mathcal{H}_n \Rightarrow A\psi \in \mathcal{H}_n$$

b) For each n, there is a $\phi_n \in \mathcal{H}_n$ which is cyclic for $A|_{\mathcal{H}_n}$; <u>ie</u>.

$$\mathcal{H}_n = \overline{\{ f(A)\phi_n \,/\, f \in C(\sigma(A)) \}}$$

Combining Lemma 1 & 2 we now have the form of the spectral theorem is most useful / transparent : multiplication form

Theorem VII. 3 Let A be bdd s.a. on $\mathcal{H}$ separable. Then $\exists$ measures $\{\mu_n\}_{n=1}^{N}$ ($N = 1, 2, \dots$ or ∞) on $\sigma(A)$ and a unitary operator $U : \mathcal{H} \longrightarrow \bigoplus_{n=1}^{N} L^2(\mathbb{R}, d\mu_n)$

so that $(UAU^{-1}\psi)_n(\lambda) = \lambda \psi_n(\lambda)$

where we write an element $\psi \in \bigoplus_{n=1}^{N} L^2(\mathbb{R}, d\mu_n)$ as an N-tuple $\langle \psi_1(\lambda), \dots, \psi_N(\lambda) \rangle$.

This realization of A is called spectral representation

Proof: Use Lemma 2 to find the decomposition and apply Lemma 1 to each component.

Remark: Theorem VII.3 says that every bdd s.a. op. is a multiplication operator on a suitable msr. space; what changes as the operator changes is the underlying measures. More precisely we have :

Corollary: Let A be bdd s.c. on seperable $\mathcal{H}$. Then, $\exists$ a finite measure space (M, μ), a bounded function F on M and a unitary map $U : \mathcal{H} \longrightarrow L^2(M, d\mu)$ /

$$\left(UAU^{-1}f\right)(m) = F(m) f(m).$$

Proof: Choose the cyclic vectors ϕ_n so that $\|\phi_n\| = 2^{-n}$. Let $M = \bigcup_{n=1}^{N} \mathbb{R}$, ie the union of N copies of $\mathbb{R}$. Define μ by requiring that its restriction to the n^{th} copy of $\mathbb{R}$ be μ_n. Since $\mu(M) = \sum_{n=1}^{N} \mu_n(\mathbb{R}) < \infty$, μ is finite. #

Definition: The measures $d\mu_n$ are called spectral measures, they are just $d\mu_\psi$ for suitable ψ.

These measures are NOT uniquely determined.

Examples: ① Let A be compact and s.a. The Hilbert-Schmidt theorem tell us that $\exists$

a complete o.n. set of eigenvectors $\{\psi_n\}_{n\geq 1}$
with $A\psi_n = \lambda_n \psi_n$.
If there is no repeated eigenvalue, then
$\sum_{n=1}^{\infty} 2^{-n}\,\delta(x-\lambda_n)$ works as spectral measure.

(2) Consider $\frac{1}{i}\frac{d}{dx}$ $\left(= -i\frac{d}{dx}\right)$ on $L^2(\mathbb{R}, dx)$. This is an unbounded s.a. operator and hence not strictly within the context we just discussed, but there is an analogue of Theorem VII.3 for this case (Chapter VIII Sections 3)

We thus seek an operator U and a measure $d\mu$ with $U : L^2(\mathbb{R}, dx) \to L^2(\mathbb{R}, d\mu(k))$ so that

(only 1 msr is needed here)

$$U\left(\frac{1}{i}\frac{d}{dx} f\right)(k) = k\,(Uf)(k)$$

The Fourier transform $(Uf)(k)$ is defined as $\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)\, e^{-ikx}\, dx$ [redacted]

[redacted] works in this case $\Rightarrow$ spectral representation in this case.

Spectral Measures and Spectrum

Definition: If $\{\mu_n\}_{n=1}^{N}$ is a family of measures then the support of $\{\mu_n\}$ is the complement of the largest open B with $\mu_n(B) = 0 \ \forall n$

so $\quad \operatorname{supp}\{\mu_n\} = \overline{\bigcup_{n=1}^{N} \operatorname{supp} \mu_n}$

Proposition: Let A be a self-adjoint op. and $\{\mu_n\}_{n=1}^{N}$ a family of spectral measures.

Then, $\quad \sigma(A) = \operatorname{supp}\{\mu_n\}_{n=1}^{N}$.

Description of $\sigma(A)$ in terms of the more general multiplication op. (Th. VII.3)

Definition: Let F be a real valued function on a measure space (M, μ). We say that

$\lambda \in$ essential range of $F \iff$

$$\mu(\{m \,/\, \lambda - \varepsilon < F(m) < \lambda + \varepsilon\}) > 0$$

$\forall \varepsilon > 0$.

Proposition: Let F be a bounded real-valued function on a measure space (M, μ). Let T_F be the operator on $L^2(M, \mu)$ given by

$$(T_F\, g)(m) = F(m).\, g(m).$$

Proof. Pb. 17b: Then $\sigma(T_F)$ is the essential range of F.

Remarks ① A unitary invariant of a s.a. operator A is a property P / $P(A) = P(UAU^{-1})$ for all unitary operators U. Thus, unitary invariant are intrinsic properties of a s.a. operator, ie. independent of "representation". An example of such invariant is the spectrum $\sigma(A)$.

② But very different op. might have same spectrum so σ is not such a good invariant

Ex: • Multiplication by x on $L^2([0,1], dx)$

• Operator with a complete set of eigenfunctions having all rationals on $[0,1]$ as eigenvalues.

Both have same spectrum, namely $[0,1]$ (!)

This motivates the following decomposition (namely finding better invariants.) of spectral measures (before passing to supports).

Recall: any measure μ on $\mathbb{R}$ has a unique decomposition into

$$\mu = \mu_{pp} + \mu_{ac} + \mu_{sing}$$

μ_{pp}: pure point msr; μ_{ac}: a.c. w/r/t Lebesgue; μ_{sing} → continuous and singular w/r/t Lebesgue

These three pieces are mutually singular so

$$L^2(\mathbb{R}, d\mu) = L^2(\mathbb{R}, d\mu_{pp}) \oplus L^2(\mathbb{R}, d\mu_{ac}) \oplus L^2(\mathbb{R}, d\mu_{sing})$$

Pb. 18) Remark: One can show that any $\psi \in L^2(\mathbb{R}, d\mu)$ has an absolutely continuous spectral measure $d\mu_\psi \iff \psi \in L^2(\mathbb{R}, d\mu_{ac})$; and similarly for p.p. and sing. msrs.

If $\{\mu_n\}_{n=1}^N$ is a family of spectral measures, we can sum $\bigoplus L^2(\mathbb{R}, d\mu_{n,ac})$ by

DEFINITION: Let A be bdd s.a. operator on $\mathcal{H}$. Let $\mathcal{H}_{pp} := \{\psi \,/\, \mu_\psi \text{ is p.p.}\}$; $\mathcal{H}_{ac} := \{\psi \,/\, \mu_\psi \text{ is a.c.}\}$; $\mathcal{H}_{sing} := \{\psi \,/\, \mu_\psi \text{ is continuous singular}\}$

All in all

Theorem VII.4: $\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_{ac} \oplus \mathcal{H}_{sing}$

Each of these subspaces is invariant under A. $A/_{\mathcal{H}_{pp}}$ has a complete set of eigenvectors, $A/_{\mathcal{H}_{ac}}$ has only a.c. spectral measures and $A/_{\mathcal{H}_{sing}}$ has only continuous singular spectral measures.

Definition: $\sigma_{pp}(A) := \{\lambda / \lambda \text{ is an eigenvalue of } A\}$

$\sigma_{cont} := \sigma(A/_{\mathcal{H}_{cont} = \mathcal{H}_{sing} \oplus \mathcal{H}_{a.c}})$

$\sigma_{ac} := \sigma(A/_{\mathcal{H}_{ac}})$

$\sigma_{sing} := \sigma(A/_{\mathcal{H}_{sing}})$

These sets are called the PURE POINT, CONTINUOUS, ABSOLUTE CONTINUOUS and singular spectrum respectively. (or continuous singular)

Remark: Note that we defined above σ_{pp} as the actual set of

eigenvalues and not as $\sigma(A/_{\mathcal{H}_{pp}})$.
As a consequence, it may happen that
$\sigma_{ac} \cup \sigma_{sing} \cup \sigma_{pp} \neq \sigma$

Remark: σ_{cont} is different from what in other literature is refered to as "continuous spectrum" (defined as the set of elements in $\sigma(A)$ which are neither in the point spectrum nor in the residual spectrum)
To illustrate the difference consider the s.a operator A on $\mathcal{H} = \mathbb{C} \oplus L^2[0,1]$

$$A(z, f(x)) = \left(\tfrac{1}{2} z, x f(x)\right)$$

With our definition $\frac{1}{2} \in$ ~~p~~ p.p and cont.
Other literature however assign $\frac{1}{2}$ to p.p.
and their "continuous spectrum" is $[0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$

Proposition: $\sigma_{cont} = \overline{\sigma_{ac}(A) \cup \sigma_{sing}(A)}$
A s.a on $\mathcal{H}$.
$\sigma(A) = \overline{\sigma_{pp}(A)} \cup \sigma_{cont}(A)$

- The sets however need not be disjoint.
- $\sigma_{sing}(A)$ may have nonzero Lebesgue measure.