Handout M523H

Section 2.1

Problem 3d)

Prove that the sequence $a_n = \frac{2^n}{n!}$ converges by letting $\varepsilon > 0$ be given and finding $N = N(\varepsilon)$ so that $|a_n - a| \leq \varepsilon$ for all $n \geq N$.

First note that $2^n \leq n!$ for all $\underline{n \geq 4}$ and moreover note that

$$3^n \leq n! \quad \text{for all } \underline{n \geq 7}$$

(You can prove both of these facts by induction on n). Thus $\frac{2^n}{n!} \underset{(\text{if } n \geq 7)}{\leq} \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n \xrightarrow[n\to\infty]{} 0$ $\underline{\text{as } n \to \infty}$

So now let's prove formally (ie. by definition) that the limit is indeed 0.

Given $\varepsilon > 0$ we must choose $N = N(\varepsilon)$ so that $\left|\frac{2^n}{n!} - 0\right| \leq \varepsilon$ for all $n \geq N$.

So let $\varepsilon > 0$ be given, we have

$$\left| \frac{2^n}{n!} - 0 \right| = \left| \frac{2^n}{n!} \right| = \frac{2^n}{n!} \leq \frac{2^n}{3^n}$$

provided $n \geq 7$.

Now, $\frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n \leq \varepsilon$ if and only if

$$n \ln(2/3) \leq \ln \varepsilon$$

$$\Rightarrow \quad n \geq \frac{\ln \varepsilon}{\ln(2/3)} = \frac{\ln(1/\varepsilon)}{\ln(3/2)}$$

(Note. $\ln \varepsilon$ and $\ln(2/3)$ are NEGATIVE).

Hence if we let $N = \max\left\{7, \frac{\ln(1/\varepsilon)}{\ln(3/2)}\right\}$

we have that

$$\left| \frac{2^n}{n!} - 0 \right| \leq \varepsilon \quad \text{for all } n \geq N.$$

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Problem 4 d)

Prove that $a_n = \frac{n!}{2^n}$ converges to ∞.

Given M we must choose N so that

$n \geq N$ implies that $a_n \geq M$.

So let M be given ; we have

$$a_n = \frac{n!}{2^n} \geq \frac{3^n}{2^n} \quad \text{provided } n \geq 7$$

Moreover $\frac{3^n}{2^n} = \left(\frac{3}{2}\right)^n \geq M$ if and

only if $n \ln(3/2) \geq \ln M$

$$\Rightarrow \quad n \geq \frac{\ln M}{\ln(3/2)}$$

Hence if we choose $N = \max\left\{7, \frac{\ln M}{\ln(3/2)}\right\}$

we have that

$$a_n = \frac{n!}{2^n} \geq M \quad \text{for all } n \geq N$$

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