QUICK REVIEW OF IMPORTANT RESULTS FROM M623/M624 CONTAINED IN CH. 4 (BRÉZIS)

Recall: (X, Σ, μ) a σ-finite measure space

Then $L^p(X)$, $1 \le p < \infty$ is defined as

$$\left\{ f \text{ measurable}, f: X \to \mathbb{R} \text{ or } \mathbb{C} \;/\; \int_X |f(x)|^p \, d\mu < \infty \right\}$$

Then $\|f\|_{L^p(X)} := \left(\int_X |f(x)|^p \, d\mu \right)^{1/p}$ is a norm and $L^p(X)$ is a normed v.s.

Furthermore $L^p(X)$ is complete under $\|\cdot\|_p$ hence $L^p(X)$ is a Banach space.

Case $p = \infty$:

$$L^\infty(X) = \left\{ f : X \to \mathbb{R} \text{ or } \mathbb{C}, \text{ measurable} \;/\; \operatorname*{ess\,sup}_{x \in X} |f(x)| < \infty \right\}$$

$\operatorname*{ess\,sup}_{x \in X} |f(x)| < \infty$ means that given f, $\exists\, C_f > 0$ s.t. $|f(x)| < C_f$ a.e. $x \in X$

$$\|f\|_{L^\infty} = \operatorname*{esssup}_{x\in X} |f(x)|$$

$$= \inf\{C_f > 0 \ / \ |f(x)| \leq C_f \text{ a.e } x\in X\}$$

$\|\cdot\|_{L^\infty}$ is also a norm. ($|f(x)| \leq \|f\|_{L^\infty}$ a.e $x\in X$)

and L^∞ is complete under $\|\cdot\|_{L^\infty}$.

So L^∞ is also Banach. All in all we have

(Brézis Ch. 4)

Theorem 4.8 (Fisher-Riesz): For $1 \leq p \leq \infty$ $L^p(X)$ is a Banach space.

Theorem 4.9: Let $\{f_n\}_{n\geq 1}$ be a sequence in L^p and let $f \in L^p$ be such that $f_n \to f$ in L^p (i.e. $\|f_n - f\|_{L^p} \to 0$ as $n\to\infty$). Then there exists a subsequence $\{f_{n_k}\}_{k\geq 1}$ and a function $h \in L^p$ such that :

(a) $f_{n_k}(x) \to f(x)$ a.e. $x \in X$

(b) $|f_{n_k}(x)| \leq h(x)$ $\forall k$ a.e $x\in X$.

- Next we discuss reflexivity/Seperability/Duality of L^p

We divide our discussion to 3 cases

(A) $1<p<\infty$

(B) $p=1$

(C) $p=\infty$

The space $L^p(X)$, $1<p<\infty$ is reflexive, separable and the dual of L^p is $L^{p'}$ with $\frac{1}{p}+\frac{1}{p'}=1$

That is $(L^p)^* = L^{p'}$.

- Reflexivity of L^p $1<p<\infty$ is in Theorem 4.10

Note that if $1<p<\infty \Rightarrow 1<p'<\infty$ and so $(L^{p'})^* = L^p$ $\quad \frac{1}{p'}+\frac{1}{p}=1$, $\quad (\dagger)$

Hence $\quad (L^p)^{**} = L^p \quad 1<p<\infty$.

$(\dagger)$ uses the duality identification given by

Theorem 4.11 (Riesz Representation Theorem)

Let $1<p<\infty$ and let $\phi \in (L^p)^*$ — that is ϕ is a continuous linear functional on L^p —

Then there exists a UNIQUE FUNCTION $u \in L^{p'}$,

$(1<p'<\infty)$ $\frac{1}{p}+\frac{1}{p'}=1$ such that

$$\langle \phi, f\rangle = \int_X u f \quad \forall f \in L^p$$

Moreover, $\|u\|_{L^{p'}} = \|\phi\|_{(L^p)^*}$.

Remark: This theorem is of fundamental importance. It says that every continuous linear functional on L^p $(1<p<\infty)$ can be represented as an integral against a function $u \in L^{p'}$ $\left(\frac{1}{p}+\frac{1}{p'}=1\right)$. The mapping

$$\Phi \longmapsto u \quad \text{is linear and}$$

a surjective isometry which allow us to identify the abstract dual space of L^p with $L^{p'}$ $\left(\frac{1}{p}+\frac{1}{p'}=1\right)$. $p' :=$ dual exponent

One then uses this identification systematically that is uses $(L^p)^* = L^{p'}$.

Theorem 4.12: Consider $X = \mathbb{R}^d$. The space $C_c(\mathbb{R}^d)$ of continuous functions with compact support in $\mathbb{R}^d$ is dense in L^p for $1 < p < \infty$ and ALSO for $p = 1$.

[In fact for $1 \le p < \infty$ the set of SIMPLE functions $\varphi = \sum_{j=1}^{N} a_j \chi_{E_j}$ where E_j is measurable with $m(E_j) < \infty \ \forall j$ is dense in L^p]

Theorem 4.13: $L^p(\mathbb{R}^d)$ is separable for any $1 \le p < \infty$.

Remark: The separability of L^p $1 \le p < \infty$ holds more generally for measure spaces X which themselves are separable (in the sense that $\exists$ a countable family of open sets in Σ s.t. the σ-algebra generated by these coincides with Σ. (ie. Σ is the smallest σ-algebra containing such countable family of open sets.),

(X, Σ, μ)

Theorem 4.14 (Riesz Representation theorem for $L^1(X)$)

Let $\phi \in (L^1)^*$ be a linear continuous functional on L^1. Then, there exists a unique function $u \in L^\infty(X)$ / $\langle \phi, f \rangle = \int_X u f \quad \forall f \in L^1$

Moreover,

$$\|u\|_{L^\infty} = \|\phi\|_{(L^1)^*}$$

As before, $\phi \mapsto u$ is a linear surjective isometry and we systematically identify $(L^1)^* = L^\infty$.

Remark: $L^1(X)$ is NEVER reflexive.

(except if X is finite set, $L^1(X)$ finite dim). ("trivial" case).

- We prove this when $X = \mathbb{R}^d$, $\mu = m$ (Lebesgue msr.)

We ARGUE BY CONTRADICTION:

Consider a sequence of sets $E_n \supseteq E_{n+1} \ldots$ / $m(E_n) > 0 \ \forall n$ and $m(E_n) \to 0$

One can always construct this by seq. of Lebesgue msr on $\mathbb{R}^d$

Let $h_n = \chi_{E_n}$ and define $u_n = \dfrac{h_n}{\|h_n\|_{L^1}}$

Since $\|u_n\|_{L^1} = 1 \ \forall n$, $\exists$ a subsequence u_{n_k} / $u_{n_k} \xrightarrow[k \to \infty]{} u$ in weak top $\sigma(L^1, L^\infty)$

Using contrad. hypothesis of reflexivity of L^1

by Theorem 3.18, ie.

$$\int u_{n_k} \phi \xrightarrow{k \to \infty} \int u \phi \quad \forall \phi \in L^\infty$$

On the other hand, for any fixed j and $n_k > j$ we have that $\int u_{n_k} \cdot h_j = 1$

(Recall $n_k > n_{k-1}$ so by letting $k \to \infty$ $\exists$ $n_k > j$).

$$\downarrow k \to \infty$$

$$\int u \cdot h_j = 1 \quad \forall j$$

On the other hand, by the dominated convergence theorem, $\int u \, h_j \to 0$ as $j \to \infty$. which is a contradiction.

- We prove that $\ell^1(\mathbb{N})$ is not reflexive. Consider

$e_n = (0, 0, \ldots \underset{n^{th}}{1}, 0 \ldots 0 \ldots)$.

Assume ℓ^1 is reflexive. Then $\exists$ a subsequence (e_{n_k}) and some $x \in \ell^1$ / $e_{n_k} \longrightarrow x$ in the weak topology $\sigma(\ell^1, \ell^\infty)$; ie.

$$\langle \varphi, e_n \rangle \longrightarrow \langle \varphi, x \rangle \quad \forall \varphi \in \ell^\infty$$

Choosing $\varphi = \varphi_j = (0, \ldots, 0, \underset{(j)}{1}, 1, 1, 1 \ldots)$ we find that $\langle \varphi_j, x \rangle = 1 \ \forall j$. On the other hand, $\langle \varphi_j, x\rangle \to 0$ as $j \to \infty$ (since $x \in \ell^1$)

contradiction.

(c) We finally turn our attention to L^∞. Recall $L^\infty = (L^1)^*$ so L^∞ is a "dual space" and hence in particular we have

i) The closed unit ball B_{L^∞} is compact in the weak* topology $\sigma(L^\infty, L^1)$ (by Theorem 3.16).

ii) If $\Omega \subset \mathbb{R}^d$ is a measurable subset and (f_n) is a bounded sequence in $L^\infty(\Omega)$, then there exists a subsequence $(f_{n_k})_{k \geq 1}$ and some $f \in L^\infty(\Omega)$ / $f_{n_k} \to f$ in the weak* topology $\sigma(L^\infty, L^1)$ (as a consequence of Corollary 3.30 and Theorem 4.13)

- However (!) $L^\infty(\Omega)$ is NOT reflexive. (except in trivial case Ω = finite set). Furthermore as we discussed before (while in Section 3-NOTES) $(L^\infty)^* \supsetneq L^1$.