Handout Math 523: Problem 9 Section 1.3

9c) The answer is uncountable. To prove this let's just consider the case of two elements. That is let $A = \{ f : f : \mathbb{N} \to \{0,1\} \}$.

Claim: A is uncountable.

There are several proofs of this. The "shortest" one is as follows. Assume A is countable. Then there is a one-to-one correspondence $F : \mathbb{N} \to A$. Since each $f \in A$ can be thought of as a sequence of 0 and 1 we can write: $F(n) = (x_{n_1}, x_{n_2}, x_{n_3}, \ldots)$ for all $n \in \mathbb{N}$.
(where each infinite-tuple on the right hand side corresponds to an f in A. ($x_{n_j} = 0$ or 1 $\forall j \, \forall n$))

Now define $y_n = 1 - x_{n_n}$ $\forall n \in \mathbb{N}$.

Then $\left\{ \begin{array}{l} y_n = 1 \text{ if } x_{n_n} = 0 \\ y_n = 1 \text{ if } x_{n_n} = 1 \end{array} \right\} \Rightarrow$ $y = (y_n)$ defines a function: $\mathbb{N} \to \{0,1\}$ BUT $y \notin \operatorname{Ran} F$ contrad

9d) The answer is countable. To prove this one first follows the hint and then applies problem 8) with

A_n = set of all subsets of $\mathbb{Z}$ with cardinality n.

- To establish the hint note that for each fixed $n \in \mathbb{N}$ there is a one-to one function between

$$A_n \longrightarrow \underbrace{\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}}_{n \text{ copies of } \mathbb{N}}$$

Since each set E in A_n (ie $E \subseteq \mathbb{Z}$, $\text{card}(E) = n$) can be uniquely assigned to an n-tuple each entry of which corresponds to an element in E. Hence $\text{card}(A_n) \leq \text{card}(\underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{n\text{-times}}) = \text{card}(\mathbb{N})$

But since A_n is infinite $\text{card}(A_n) = \text{card}(\mathbb{N})$.