Handout Math 523H

Section 2.2.

Problem 5 (Theorem 2.2.6)

Let $\{a_n\}$ and $\{b_n\}$ be sequences and suppose $a_n \to a$ and $b_n \to b$ as $n \to \infty$.

Suppose $b \neq 0$ and $b_n \neq 0$ for any n. Then prove that $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{a}{b}$

Proof: We follow the hint and first prove that $\exists\ M > 0$ such that

$$0 < M \leq |b_n| \quad \text{for all } n$$

(NOTE: the upper bound is not needed for this problem)

- Indeed, since $b_n \to b$ there exists N_1 such that $|b_n - b| \leq \dfrac{|b|}{2}$ for all $n \geq N_1$

Hence if $n \geq N_1$

$$|b_n| = |b_n - b + b| \geq |b| - |b_n - b| \geq \frac{|b|}{}$$

Since $b_n \neq 0$ for all n let

$$M = \min\left\{ |b_1|, |b_2|, \ldots, |b_{N_1-1}|, \frac{|b|}{2} \right\}$$

Then we have that $|b_n| \geq M$ for all $n \geq 1$

- Now let us prove that $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$

Given $\varepsilon > 0$ we must choose $N = N(\varepsilon) > 0$

so that for $n \geq N$ $\left| \frac{a_n}{b_n} - \frac{a}{b} \right| < \varepsilon$

So let $\varepsilon > 0$ be given ; we write

$$\left| \frac{a_n}{b_n} - \frac{a}{b} \right| = \left| \frac{a_n}{b_n} - \frac{a_n}{b} + \frac{a_n}{b} - \frac{a}{b} \right|$$

$$\leq |a_n| \left| \frac{1}{b_n} - \frac{1}{b} \right| + \frac{1}{|b|} |a_n - a|$$

$$= |a_n| \frac{|b_n - b|}{|b_n|.|b|} + \frac{1}{|b|} |a_n - a|$$

(Note that by hypothesis $b_n \neq 0$ for all n and $b \neq 0$)

Consider the first term. ~~Since~~ $a_n \to a$ by Proposition 2.2.1 we have that a_n is BOUNDED.

That is there exists some constant $A > 0$ such that $|a_n| \leq A$ for all n.

On the other hand we have proved at the begining that $\exists\, M > 0$ such that $|b_n| \geq M$ for all n. Therefore we have that

$$|a_n| \frac{|b_n - b|}{|b_n||b|} \leq \frac{A}{M|b|} |b_n - b| .$$

(Note that $\dfrac{A}{M|b|}$ is a CONSTANT indep. of n).

But since $b_n \to b$ given $\varepsilon > 0$ we can find N_1 such that $|b_n - b| < \dfrac{\varepsilon}{2}\left(\dfrac{M|b|}{A}\right)$ for all $n \geq N_1$.

For the second term we choose N_2 so that $|a_n - a| < \frac{|b|\varepsilon}{2}$ for $n \geq N_2$

which can be done since $a_n \to a$.

Finally let $N = \max\{N_1, N_2\}$, then for all $n \geq N$ we have that

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| \leq |a_n| \frac{|b_n - b|}{|b_n||b|} + \frac{1}{|b|}|a_n - a|$$

$$\leq \frac{A}{M|b|}|b_n - b| + \frac{1}{|b|}|a_n - a|$$

$$\leq \frac{\cancel{A}}{M\cancel{|b|}} \frac{\varepsilon}{2}\left(\frac{M|b|}{A}\right) + \frac{1}{\cancel{|b|}}\frac{\varepsilon}{2}\cancel{|b|}$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

as desired. #