Handout about the first part of the proof of Theorem 3.4 (that is proof of (ii) $D^+(F)(x) \leq D_-(F)(x)$ (a.e x) in the case when F is assumed to be increasing, bounded and continuous (extra assumption that we'll remove in Section 3.3)).

→ To prove then that $D^+(F)(x) \leq D_-(F)(x)$ a.e. x we consider $R > r$ real numbers and let

$$E_{R,r} := \{ x \in [a,b] : D^+(F)(x) > R \text{ and } D_-(F)(x) < r \}$$

$\parallel$

E (abbreviate notation).

If we prove that $m(E) = 0$ then by varying R, r over all rationals (with $R > r$) and taking the (countable) union of all the corresponding $E_{R,r}$ we'll obtain again a set of measure zero and have thus shown that $D^+(F)(x)$ must be $\leq D_-(F)(x)$ for almost all x.

→ To prove then that $m(E) = 0$ ($R > r$ fixed) we assume that $m(E) > 0$ and derive a contradiction.

Since $R/r > 1$ $\exists$ an open set $\mathcal{O}$ such that $E \subset \mathcal{O} \subset (a,b)$ and $m(\mathcal{O}) < m(E) \cdot \frac{R}{r}$

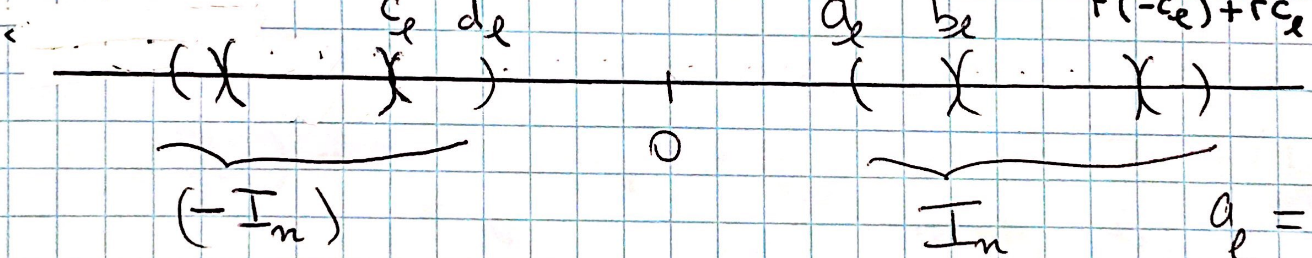
Since $\mathcal{O}$ is open we can write $\mathcal{O} = \bigcup\limits_{n \geq 1}^{D} I_n$ where I_n are open intervals. Fix n. We want to apply Corollary 3.6 to

$$\left[\begin{array}{l} G : (-I_n) \longrightarrow \mathbb{R} \\ G(x) := F(-x) + rx \\ \qquad x \in (-I_n). \end{array}\right.$$

We get then $\exists$ $\mathcal{E}_n \subset (-I_n)$ (open) $\mathcal{E}_n = \bigcup\limits_{\ell \geq 1}^{D} J_\ell$

$J_\ell = (c_\ell, d_\ell)$ open intervals and $G(c_\ell) \leq G(d_\ell)$

$F(-c_\ell) + rc_\ell \leq F(-d_\ell) + rd_\ell$



$a_\ell = -d_\ell$

$b_\ell = -c_\ell$

Reflecting $J_\ell = (c_\ell, d_\ell)$ and $(-I_n)$ through the origin we then get $\exists$ a_ℓ, b_ℓ /

$\bigcup\limits_{\ell \geq 1} (a_\ell, b_\ell) \subset I_n$ and

$$\boxed{F(b_\ell) - F(a_\ell) \leq r(b_\ell - a_\ell)} \quad (+)$$

③

Now on each (a_ℓ, b_ℓ) we apply Corollary 3.6 again but now to $\left\{ \begin{array}{l} G : (a_\ell, b_\ell) \to \mathbb{R} \\ G(x) := F(x) - R.x \end{array} \right.$

$\Rightarrow$ $\exists$ an open (which we write as) $\bigcup_{j\geq 1}^{D} (a_{\ell,j}, b_{\ell,j}) \subset (a_\ell, b_\ell)$

and $G(a_{\ell,j}) \leq G(b_{\ell,j}) \Leftrightarrow$

$$F(a_{\ell,j}) - R\, a_{\ell,j} \leq F(b_{\ell,j}) - R\, b_{\ell,j}$$

$$\Leftrightarrow \quad \boxed{F(b_{\ell,j}) - F(a_{\ell,j}) \geq R\,(b_{\ell,j} - a_{\ell,j})} \quad (\dagger\dagger)$$

Let now $\mathcal{O}_n = \bigcup_{\ell}^{D} \underbrace{\bigcup_{j}^{D} (a_{\ell,j}, b_{\ell,j})}_{(a_\ell, b_\ell)}$, we have:

$$m(\mathcal{O}_n) = \sum_\ell \sum_j (b_{\ell,j} - a_{\ell,j}) \underset{(\dagger\dagger)}{\leq} \frac{1}{R} \sum_{\ell,j} F(b_{\ell,j}) - F(a_{\ell,j})$$

Use that F is increasing

$$\leq \frac{1}{R} \sum_\ell F(b_\ell) - F(a_\ell)$$

$$\underset{(\dagger)}{\leq} \frac{r}{R} \sum_\ell (b_\ell - a_\ell) \leq \frac{r}{R} m(I_n)$$

Note that $\mathcal{O}_n \supset E \supset I_n$ since $D^+F(x) > R$ and $r > D_-F(x)$ for $x \in E$. And $I_n \supset \mathcal{O}_n$ also. So

$$m(E) = \sum_n m(E \cap I_n) \leq \sum_n m(\mathcal{O}_n) \leq \frac{r}{R} \sum_n m(I_n) = \frac{r}{R} m(\mathcal{O}) < m(E)$$

CONTRADICTION ($m(E) < m(E)$!)