M725. Section 3.5 . Reflexive Spaces
(Brezis's book)

Definition: E Banach, $J: E \to E^{**}$ the canonical injection from E to E^{**}. ($E \subset E^{**}$)
We say that E is reflexive if J is surjective; in which case $J(E) = E^{**}$ and we then identify E^{**} with E (i.e. $E = E^{**}$)

(Remark: If E is finite dim. then $E = E^{**}$ by dim. counting)

Examples For $1 < p < \infty$ L^p is reflexive

$(L^p)^* = L^{p'}$ and $\frac{1}{p} + \frac{1}{p'} = 1$

$(L^{p'})^* = L^p$

$1 < p' < \infty$ $\quad (L^p)^{**} = L^p$

However: L^1 and L^∞ are not reflexive.

$$(L^1)^* = L^\infty \quad \text{but} \quad (L^\infty)^* \neq L^1 \subset (L^\infty)^*$$

$L^\infty(X, \Sigma, \mu)$

(σ-finite msr space)

$(L^\infty)^* = ba(X, \Sigma, \mu)$ = bdd additive measures = space of finitely additive (signed) measures on Σ which are absolutely continuous w.r.t. μ, equipped with the total variation norm.
(see Dunford-Schwartz, Vol. I).

Let's see that indeed $(L^\infty)^*$ is strictly bigger than L^1: $\underline{\text{WTS.}}$ $\exists$ continuous linear functionals on L^∞ which cannot be represented as

$$\underbrace{\phi(f)}_{=} \langle \phi, f \rangle = \int f\, u\, dx \quad \forall f \in L^\infty \text{ and some } u \in L^1.$$

(Riesz Rep Theorem)

Let's see a specific example of such functional. Consider $\psi: C_c^\infty(\mathbb{R}^d) \to \mathbb{R}$ defined by

$$\psi(f) = f(0) \quad \text{for } f \in C_c^\infty$$

ψ is linear and continuous on C_c^∞ for the $\|\cdot\|_\infty$ norm. By the HB theorem we may extend ψ into a linear continuous ~~linear~~ functional ϕ on $L^\infty(\mathbb{R}^d)$ and have then

$$(*) \quad \langle \phi, f \rangle = f(0) \quad \forall f \in C_c^\infty$$

$\underline{\text{By contradiction}}$ Assume $\exists\, u \in L^1 \,/\, \langle \phi, f \rangle = \int u f \quad \forall f \in L^\infty$

In particular this must hold for all $f \in L^\infty \,/\, f(0) = 0 \Rightarrow \int_{\mathbb{R}^d} u f = 0$ for all those f. But then $u \equiv 0$ a.e in $\mathbb{R}^d$ (since $u \equiv 0$ a.e in $\mathbb{R}^d \setminus \{0\}$), $\Rightarrow \langle \phi, f \rangle \equiv 0 \;\forall f \in L^\infty$

Contradicting $(*)$

Theorem 3.17 (Kakutani): Let E be Banach. Then E is reflexive if and only if

$$B_E = \{x \in E \,;\ \|x\| \le 1\} \text{ is compact}$$

in the weak topology $\sigma(E, E^*)$.

$\Rightarrow)$

Proof: Assume E is reflexive, so that $J(B_E) = B_{E^{**}}$. By Theorem 3.16, $B_{E^{**}}$ is compact in the topology $\sigma(E^{**}, E^*)$. So we need to show that J^{-1} is continuous from E^{**} equipped with $\sigma(E^{**}, E^*)$ to E equipped with $\sigma(E, E^*)$. We invoke once again Prop 3.2, to deduce this from proving that $\forall f \in E^*$ fixed, the map $\xi \longmapsto \langle f, J^{-1}\xi\rangle$ is continuous on E^{**} equipped with $\sigma(E^{**}, E^*)$. But $\langle f, J^{-1}\xi\rangle = \langle \xi, f\rangle$ and the map $\xi \longmapsto \langle \xi, f\rangle$ is continuous on E^{**} in the $\sigma(E^{**}, E^*)$ topology. Hence B_E is compact in $\sigma(E, E^*)$.

$\Leftarrow)$ This is harder !! needs Lemma 3.3 (Helly)
See Brezis pages 68–69. Lemma 3.4 (Goldstine)

Theorem 3.18: Assume E is reflexive Banach and let $(x_n)_{n\geq 1}$ be a bounded sequence in E. Then $\exists$ a subsequence $(x_{n_k})_{k\geq 1}$ that converges in the weak topology $\sigma(E, E^*)$.

and conversely **Theorem 3.19**: Assume that E is a Banach space / every bdd sequence admits a weakly convergent subsequence (in $\sigma(E, E^*)$ topology) Then, E is reflexive.

Remark: • In a metric compact space X every sequence in X admits a convergent subsequence. (in fact in metric spaces the latter is $\Leftrightarrow$ compact).

• Furthermore, $\exists$ compact topological spaces X and some sequences in X without any convergent subsequence. Example: $X = B_{E^*}$ which is compact in $\sigma(E^*, E)$, when $E = \ell^\infty$ one may construct a sequence in X without any convergent subsequence.

• If X is a topological space, having the property that every sequence admits a convergent subsequence $\not\Rightarrow$ X is compact !

PROPOSITION 3.20 : Assume that E is a reflexive Banach space and let $M \subset E$ be a closed linear subspace of E. Then M is reflexive.

Remark : M equipped with the norm of E has a priori 2 weak topologies :

a) The one induced by $\sigma(E, E^*)$

b) its own weak topology $\sigma(M, M^*)$.

But in fact these two topologies are the same by the HB theorem. (think about this).

To prove the proposition we can use Theorem 3.17.

That is WTS B_M is compact in the $\sigma(M, M^*)$ topology $\Leftrightarrow$ in $\sigma(E, E^*)$ topology. Since B_E

(E reflexive Th. 3.17) is compact in $\sigma(E, E^*)$ and M is closed in

(Theorem 3.7) $\sigma(E, E^*)$ we then have B_M is compact in $\sigma(E, E^*)$.

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COROLLARY 3.21 : A Banach space E is REFLEXIBE if and only if its dual space E^* is reflexive.

Proof : Read in BRÉZIS. It's straightforward. pg 70-71

pg 72 NOTE : Also read Theorem 3.24 about unbounded linear operators $A : D(A) \subset E \to F$, $D(A)$ dense & A closed and A^* and A^{**}.

3.6 A few facts about separable spaces

- On a metric space E, separable means that $\exists\, D \subset E$ that is countable and dense
- If $F \subset E$ (metric separable) $\rightarrow$ F is separable. Indeed suppose $\{x_n\}_{n\geq 1} \subset E$ is dense and consider $(r_m)_{m\geq 1}$ a sequence of positive numbers s.t $r_m \underset{m\to\infty}{\longrightarrow} 0$. Consider $B(x_n; r_m) \cap F$. If this intersection is $\neq \emptyset$ then choose $a_{n,m} \in B(x_n, r_m) \cap F$. Clearly $\{a_{n,m}\}_{n,m\geq 1}$ is countable. And it is also dense in F (WHY?)

Read proof in Brézis pg. 73

Theorem 3.26: Let E be Banach / E^* is seperable. Then E is seperable.

CAVEAT: The converse is FALSE. Indeed consider $E = L^1$ Banach and separable. But $E^* = L^\infty$ is NOT separable

Will come back to this fact later.

Corolary 3.27: Let E be Banach. Then

E reflexive and seperable $\Longleftrightarrow$

E^* reflexive and separable.

- Separability is related to being metrizable (ie when there is a metric that induces the topology of the space).

Theorem 3.28: E seperable Banach. Then

B_{E^*} is metrizable in the weak* topology $\sigma(E^*,E)$

Conversely if B_{E^*} is metrizable in $\sigma(E^*,E)$ then E is seperable

Idea of proof of $\Rightarrow$) Consider $\{x_n\}_{n\geq 1}$ dense subset in B_E. For any $f \in E^*$ set

$$[f] = \sum_{n=1}^{\infty} \frac{1}{2^n} |\langle f, x_n \rangle|$$

- $[\cdot]$ is a norm on E^*
- $[f] \leq \|f\|_{E^*}$. Let then
- $d(f,g) := [f-g]$ d is a metric in E^*

- The topology induced by d on B_{E^*} is the same as the WEAK* topology $\sigma(E^*, E)$ restricted to B_{E^*}

Theorem 3.29 : Let E be a Banach space such that E^* is seperable. Then B_E is metrizable in the weak topology $\sigma(E, E^*)$. Conversely if B_E is metrizable in $\sigma(E, E^*)$ then E^* is separable.

Remark pg 4 03/24/2020

Recall we have seen last class that in infinite dimensions the weak topology $\sigma(E, E^*)$ (resp. weak* top $\sigma(E^*, E)$) on all of E (resp. on all of E^*) is NOT metrizable.

In particular the metric induced by the norm $[\cdot]$ on all of E^* does NOT coincide with the weak* topology.

COROLLARY 3.30 : Let E be a separable Banach space and let $\{f_n\}_{n \geq 1}$ be a bounded sequence in E^*. Then $\exists$ a subseq. $\{f_{n_k}\}_{k \geq 1}$ that converges in $\sigma(E^*, E)$.

Non separability of $L^\infty(\mathbb{R}^d)$.

(one can more generally show $L^\infty(X)$ is NOT separable except when X consists of a finite number of points).

(Brezis) **Lemma 4.2**: Let E be Banach. Assume that there exists a family $(O_i)_{i\in I}$ /

(i) For each $i \in I$, O_i is a nonempty subset of E

(ii) $O_i \cap O_j = \emptyset$ if $i \neq j$

(iii) I is uncountable.

Then E is not seperable

Proof: Suppose by contradiction E is separable and let $\{x_n\}_{n\geq 1} \subset E$ dense. For each $i \in I$ $O_i \cap (\{x_n\}_{n\geq 1}) \neq \emptyset$. Choose $n(i)$ / $x_{n(i)} \in O_i$

The map $i \longrightarrow n(i)$ is injective by (ii) $\Rightarrow$ I is countable! Contradiction. #

We use this Lemma to prove nonseparability of $L^\infty(\mathbb{R}^d)$:

Proof: ① Claim: $\exists\, (W_i)_{i\in I}$ measurable sets in $\mathbb{R}^d$ which are all distinct. That is $W_i \Delta W_j$ has positive measure if $i \neq j$ and such that I is uncountable.

The claim follows easily by considering all the balls $B(x_0, r)$ $x_0 \in \mathbb{R}^d$, $r > 0$ small enough. and arguing from here. (HOMEWORK !).

② Assuming the claim then, we can conclude from Lemma 4.2 by considering the family $(\mathcal{O}_i)_{i\in I}$ define by

$$\mathcal{O}_i = \left\{ f \in L^\infty \,/\, \|f - \chi_{W_i}\|_{L^\infty} < \frac{1}{2} \right\}$$

(NOTE: $\|\chi_{W_i} - \chi_{W_j}\|_{L^\infty} = 1$ $i \neq j$ (W_i distinct from W_j).

All in ALL in $\mathbb{R}^d$ we have

- For $1<p<\infty$ L^p is reflexive, separable, dual $L^{p'}$ $\quad \frac{1}{p} + \frac{1}{p'} = 1$
- L^1 not reflexive but yes separable, dual $L^1 = L^\infty$
- L^∞ not reflexive NOT separable, dual $L^\infty \supsetneq L^1$.